E_∞-structure for $Q_*(R)$

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Abstract

MacLane homology of a ring R is the Hochschild homology of the so called cubical construction $Q_*(R)$, which is a chain-algebra. If we take a commutative ring R, the Dixmier product on $Q_*(R)$ is no longer commutative. The main result of this paper is that it is commutative up to higher homotopies, i.e. is a E_∞-algebra. The E_∞-operad which acts on $Q_*(R)$ is constructed by using the analogue of the Q_*-complex in the context of finite sets. For the correct notation of an operad action on these complexes the definition of an E_∞-monoidal functor is introduced.

1.1 Introduction

MacLane (co)homology of a ring R with coefficients in an R-bimodul M can be defined as the Hochschild (co)homology of the differential graded ring $Q_*(R)$ with coefficients in M:

$$\begin{aligned} HML_*(R,M) &:= H_*(Q_*(R),M) \\ HML^*(R,M) &:= H^*(Q_*(R),M) \end{aligned}$$

We will recall the definition of $Q_*(R)$ below, but first we want to mention some of the remarkable properties of this (co)homology: The second MacLane cohomology $HML^2(R,M)$ classifies arbitrary extensions of R by the bimodule M, unlike the second Hochschild cohomology which classifies only split-extensions. If the ring contains the rational numbers then Hochschild-theory and MacLane-theory coincide. MacLane homology is isomorphic to stable K-theory K_*^s introduced by Waldhausen and to topological Hochschild homology THH_*. For proves of these statements see [P-W] and [D-McC].
For an arbitrary ring R the Dixmier product on the Eilenberg-MacLane cubical construction gives $Q_*(R)$ the structure of a ring. Let us recall that Hochschild homology of commutative algebras has a very rich structure with products and homology operations. As MacLane homology is a special case of Hochschild homology it would be nice to have similar structures for it. But unfortunately a commutative ring R will not lead to a commutative product in $Q_*(R)$. The main goal of this paper is to show that $Q_*(R)$ is equipped with the structure of an E_∞-algebra. Before proving this, we will introduce the notion of an E_∞-monoidal functor which describes an action of an E_∞-operad on the images of a functor between symmetric monoidal abelian categories. Instead of directly constructing an E_∞-operad that acts on $Q_*(R)$ we show that the analogue of

the Q-construction SQ_* in the set context is an E_∞- monoidal functor. The transfer of this structure to Q_* is done by using left Kan extensions.
Finally we prove that the fact that the Q-construction maps associative algebras to associative algebras and commutative ones to E_∞-algebras can be generalized to the case of A_∞- and E_∞-algebras. Applying the same methods as for proving that Q_* is an E_∞-monoidal functor yields an easy proof that the chain complex functor C_* from the category of simplicial abelian groups to the category of chain complexes is an E_∞-comonoidal functor.

The fact that $Q_*(R)$ is an E_∞-algebra for every commutative ring leads to new operations in MacLane homology. We will investigate these implications in a different paper.

1.2 Eilenberg-MacLane's cubical construction

Let us now recall the definition of the cubical construction. For a comprehensive overview see [MLa], [L], [J-P] and [F-P-S-V-W].

Given an abelian group A the Eilenberg-MacLane cubical construction assigns a chain complex $Q_*(A)$ to this group in a functorial way. This chain complex $Q_*(A)$ is a quotient of an auxiliary complex $Q'_*(A)$ which in turn is defined as the free abelian group generated by all maps from the vertices of an n-dimensional unit-cube C_n to A:

$$Q'_*(A) := \mathbf{Z}\,[A\,[C_n]]$$

Let $(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i \in \{0,1\}$ denote a vertex of the n-cube for $n > 0$ and let $()$ denote the single element of C_0. Define two cubical face maps

$$0_i, 1_i : C_n \longrightarrow C_{n+1}$$

for $1 \leq i \leq n+1$ in the following manner:

$$\begin{aligned} 0_i(\varepsilon_1, \ldots, \varepsilon_n) &= (\varepsilon_1, \ldots, \varepsilon_{i-1}, 0, \varepsilon_i, \ldots, \varepsilon_n) \\ 1_i(\varepsilon_1, \ldots, \varepsilon_n) &= (\varepsilon_1, \ldots, \varepsilon_{i-1}, 1, \varepsilon_i, \ldots, \varepsilon_n) \end{aligned}$$

Define for $1 \leq i \leq n$ the maps

$$R'_i, S'_i, P'_i : A\,[C_n] \longrightarrow A\,[C_{n+1}]$$

as follows:

$$(R'_i f)(e) := f(0_i e), \quad (S'_i f)(e) := f(1_i e), \quad (P'_i f)(e) := f(0_i e) + f(1_i e)$$

Let P_i, S_i, R_i denote the linearizations of these maps from $Q'_n(A)$ to $Q'_{n+1}(A)$. With the help of these maps we can define the boundary map for Q'_*

$$\delta := \sum_{i=1}^{n} (-1)^i (P_i - R_i - S_i)$$