DOLD-KAN TYPE THEOREM FOR Γ-GROUPS

By TEIMURAZ PIRASHVILI

A.M. Razmadze Math. Inst. Alexidze str. 1, Tbilisi,
380093. Republic of Georgia

0. Introduction

Γ-spaces were introduced by Segal [S], who proved that they are combinatorial models for connective spectra (see also [A], [BF]). Based on Kan-Thurston theorem we show that any Γ-space is stably weak equivalent to a discrete Γ-group. By a well-known theorem of Dold-Kan the Moore normalization establishes the equivalence between the category of simplicial abelian groups and the category of chain complexes (see [DP]). mimicking the construction of normalization of simplicial groups, we give a similar construction for Γ-groups. This construction is based on the notion of cross-effects of functors [BP], which is a generalizatin of the classical definition of Eilenberg and Mac Lane [EM] to the non-abelian setup. Finally a Dold-Kan type theorem for the category of Γ-groups is proved. In abelian case our theorem claims that the category of abelian Γ-groups is equivalent to the category of functors Ab^{Ω}, where Ω is the category of all nonempty finite sets and surjections. In the non-abelian case one needs to consider functors $T : \Omega \to Gr$ together with maps

$$\{,\} : T_n \times T_m \to T_{n+m}$$

satisfying additional relations. Here T_n is the value of T on the set

$$< n >: = \{1, \cdots, n\}.$$

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1. Γ-spaces and Γ-groups

1.1. Stable homotopy of Γ-spaces. Let Γ be the category of finite pointed sets. We will assume that objects of Γ are sets $[n]$, where

$$[n] := \{0, 1, ..., n\}$$

with basepoint 0, $n \geq 0$. Let $Sets_*$ be the category of all pointed sets. Let $\mathcal{C}$ be a category with zero object. A *Γ-object over $\mathcal{C}$* is a functor $A : \Gamma \to \mathcal{C}$ such that $A([0]) = 0$. Let $\mathcal{C}^\Gamma$ be the category of Γ-objects over $\mathcal{C}$. By definition a *pointed Γ-set* is the same as a Γ-object over the category of pointed sets. Similarly an *abelian Γ-group* or a *Γ-group* is the same as a Γ-object over the category of abelian groups or groups respectively. A *Γ-space* is the same as a Γ-object over the category $s.Sets_*$ of pointed simplicial sets. Hence any Γ-set (and hence any Γ-group) yields a discrete Γ-space. A natural transformation $F \to T$ of Γ-spaces is called a *pointwise fibration* (resp. *pointwise weak equivalence*) if $F([n]) \to T([n])$ is a fibration (resp. weak equivalence) for any $n \geq 0$. Any Γ-space F gives rise to a binatural transformation

$$X \wedge F(Y) \to F(X \wedge Y)$$

as follows. For any $x \in X$ define $\hat{x} : Y \to X \wedge Y$, by $y \mapsto (x, y)$. Then apply F to get $F(\hat{x}) : F(Y) \to F(X \wedge Y)$. Now one defines $X \wedge F(Y) \to F(X \wedge Y)$, by $(x, z) \mapsto F(\hat{x})(z)$. There is a standard way to prolong a Γ-space F to a functor $s.Sets_* \to s.Sets_*$. First using direct limits one can prolong F to a functor $Sets_* \to s.Sets_*$, then by degreewise action one obtains a functor from pointed simplicial sets to the category of pointed bisimplicial sets and finally by taking the diagonal of a bisimplicial set one gets a functor $s.Sets_* \to s.Sets_*$. By abuse of notation we will still denote this functor by F. By Proposition 4.9 of [BF] F preserves weak equivalences. Any Γ-space gives rise to a spectrum $F(S^n), n \geq 0$, because of the maps $S^1 \wedge F(S^n) \to F(S^{n+1})$. Hence *stable homotopy groups* of a Γ-space F are defined by

$$\pi_*^{st}(F) = colim_n \pi_{*+n} F(S^n).$$

A natural transformation $F \to T$ is called a *stable weak equivalence* if it induces isomorphisms on π_*^{st}.

1.2. Γ-groups are enough. The goal of this section is to prove that any Γ-space is stably weak equivalent to a discrete Γ-group. I learned recently that this fact was also known to Manos Lydakis.

1.2. Theorem. *Any Γ-space has the stable weak homotopy type of a discrete Γ-group.*

Proof. This is a consequence of Lemma 1.3 and Lemma 1.4.