Decomposition and l_1–embedding of weakly median graphs

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Abstract. Weakly median graphs, being defined by interval conditions and forbidden induced subgraphs, generalize quasi–median graphs as well as pseudo–median graphs. It is shown that finite weakly median graphs can be decomposed with respect to gated amalgamation and Cartesian multiplication into 5–wheels, induced subgraphs of hyperoctahedra (alias cocktail party graphs), and 2–connected bridged graphs not containing K_4 or $K_{1,1,3}$ as an induced subgraph. As a consequence one obtains that every finite weakly median graph is l_1–embeddable, that is, it embeds as a metric subspace into some $\mathbb{R}^n$ equipped with the 1–norm.

In this paper we continue to elaborate on a structure theory of graphs based on two fundamental operations, viz., Cartesian multiplication and gated amalgamation. While Cartesian multiplication is a standard operation, gated amalgamation seems to appear only in the context of median graphs and their generalizations; cf. [4, 6, 8, 23, 27]. An induced subgraph H of a graph G is called *gated* if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x'; cf. [18]. Then G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 are (isomorphic to) two intersecting gated subgraphs of G whose union is all of G. A graph with at least two vertices is said to be *prime* if it is neither a proper Cartesian product nor a gated amalgam of smaller graphs. For instance, the only prime median graph is the two–vertex complete graph K_2; see Isbell [21] and van de Vel [26]. More generally, the prime quasi–median graphs are exactly the complete graphs; quasi–median graphs were introduced by Mulder [23] and further studied in [14, 28, 8]. The pseudo–median graphs form yet another class of graphs for which the prime members are known; see Bandelt and Mulder [6]. Unfortunately, the latter class is not closed under Cartesian multiplication and does not include all quasi–median graphs. In order to overcome these deficiences we consider here the somewhat larger class of weakly median graphs, previously studied by Chepoi [10, 11] under the name "locally median" graphs. First, we say that a graph G is *weakly modular* if its shortest–path metric $d = d_G$ satisfies the following two conditions:

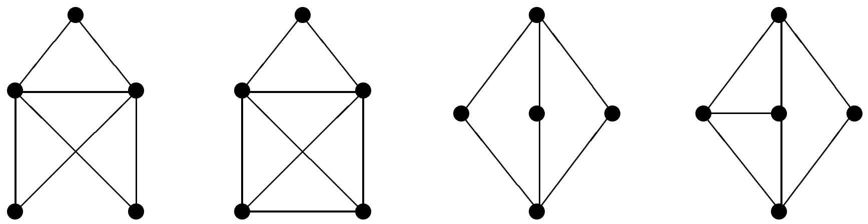
– for any three vertices u, v, w with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbour x of v and w such that $d(u,x) = d(u,v) - 1$;

– for any four vertices u, v, w, z with $d(v,z) = d(w,z) = 1$ and

$$2 = d(v,w) \leq d(u,v) = d(u,w) = d(u,z) - 1,$$

there exists a common neighbour x of v and w such that $d(u,x) = d(u,v) - 1$.

For an illustration of these conditions see [9, Fig.2]. A *weakly median* graph is a weakly modular graph that does not contain any two vertices with an unconnected triple of common neighbours; see Figure 1.



$K_{1,1,3}$ $K_{2,3}$

FIGURE 1. Forbidden induced subgraphs.

Note that all bridged graphs are weakly modular; cf. [10, 7]. A graph is called *bridged* if G does not contain any isometric cycle of length greater than 3, that is, each cycle of length greater than 3 has a shortcut in G; see Soltan and Chepoi [25] and Farber and Jamison [20]. Bridged graphs can easily be constructed since, according to [1, Corollaries 2.4 and 2.6], they admit certain vertex elimination schemes (relaxing simplicial elimination for chordal graphs). Now, a weakly median bridged graph (i.e., a bridged graph in which the first two graphs of Fig. 1 are forbidden induced subgraphs) is prime exactly when it has at least two vertices and does not have any cut vertex, that is, it is either K_2 or two–connected. This is an immediate consequence of the following two elementary facts: first, a gated subgraph cannot intersect any triangle just in a single edge, and second, in a two–connected bridged graph any two non–adjacent vertices having a common neighbour w are connected by another path within the neighbourhood of w; in consequence, two–connected bridged graphs do not have any proper gated subgraphs other than singletons. Furthermore, all wheels (whether bridged or not) are prime weakly median graphs; an *n–wheel* ($n \geq 4$) consists of a cycle of length n and a "central" vertex adjacent to all vertices of the cycle. Finally, all multipartite graphs of the form $K_{i_1,i_2,i_3,\ldots}$ (with $1 \leq i_j \leq 2$) different from $K_1, K_{1,2}$, and $K_{2,2}$ are prime weakly median graphs. A particular instance is the *α–octahedron* $K_{2,2,2,\ldots}$ (or *hyperoctahedron,* for short), which is the complement of the disjoint union of $\alpha \geq 3$ copies of K_2. For convenience, we refer to induced subgraphs of hyperoctahedra as to *subhyperoctahedra* when they contain either K_4 or an induced 4–wheel $K_{1,2,2}$ (that is, whenever they constitute 1–skeletons of at least 3–dimensional polyhedra).