SMASHING SUBCATEGORIES AND THE TELESCOPE CONJECTURE – AN ALGEBRAIC APPROACH

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Abstract. We prove a modified version of Ravenel's telescope conjecture. It is shown that every smashing subcategory of the stable homotopy category is generated by a set of maps between finite spectra. This result is based on a new characterization of smashing subcategories, which leads in addition to a classification of these subcategories in terms of the category of finite spectra. The approach presented here is purely algebraic; it is based on an analysis of pure-injective objects in a compactly generated triangulated category, and covers therefore also situations arising in algebraic geometry and representation theory.

Introduction

Smashing subcategories naturally arise in the stable homotopy category $\mathcal{S}$ from localization functors $l\colon \mathcal{S} \to \mathcal{S}$ which induce for every spectrum X a natural isomorphism $l(X) \simeq X \wedge l(S)$ between the localization of X and the smash product of X with the localization of the sphere spectrum S. In fact, a localization functor has this property if and only if it preserves arbitrary coproducts in $\mathcal{S}$. Therefore one calls a full subcategory $\mathcal{R}$ of $\mathcal{S}$ *smashing* if $\mathcal{R} = \{X \in \mathcal{S} \mid l(X) = 0\}$ for some localization functor $l\colon \mathcal{S} \to \mathcal{S}$ which preserves coproducts. In this paper we study smashing subcategories from an algebraic point of view. The main result is a new characterization of smashing subcategories which leads to a classification in terms of certain ideals in the category of finite spectra. One motivation for this work is the telescope conjecture of Ravenel and Bousfield which states that every smashing subcategory is generated by finite spectra. The approach presented here is purely algebraic and covers therefore also situations arising in algebraic geometry and representation theory where one studies certain triangulated categories having a number of formal properties in common with the stable homotopy category.

Let $\mathcal{C}$ be a compactly generated triangulated category, for example the stable homotopy category. Thus $\mathcal{C}$ is a triangulated category with arbitrary coproducts, and $\mathcal{C}$ is generated by a set of compact objects (an object X in $\mathcal{C}$ is *compact* if the representable functor $\operatorname{Hom}(X,-)$ preserves coproducts). Recall that a full triangulated subcategory $\mathcal{B}$ of $\mathcal{C}$ is *localizing* if $\mathcal{B}$ is closed under taking coproducts. We say that a localizing subcategory $\mathcal{B}$ is *strictly localizing* if the inclusion functor $\mathcal{B} \to \mathcal{C}$ has a right adjoint, and $\mathcal{B}$ is called *smashing* if there exists a right adjoint for the inclusion $\mathcal{B} \to \mathcal{C}$ which preserves coproducts. Note that a full subcategory $\mathcal{B}$ is strictly localizing if and only if there exists a localization functor $l\colon \mathcal{C} \to \mathcal{C}$ such that $\mathcal{B} = \{X \in \mathcal{C} \mid l(X) = 0\}$, and $\mathcal{B}$ is smashing if and only if the corresponding localization functor preserves coproducts.

Theorem A. *Let $\mathcal{B}$ be a localizing subcategory of $\mathcal{C}$, and denote by $\mathfrak{I}$ the ideal of maps between compact objects in $\mathcal{C}$ which factor through some object in $\mathcal{B}$. Then the following conditions are equivalent:*

(1) *$\mathcal{B}$ is smashing;*

(2) *an object X in $\mathcal{C}$ belongs to $\mathcal{B}$ if and only if every map $C \to X$ from a compact object C factors through a map $C \to D$ in $\mathfrak{I}$;*
(3) *an object X in $\mathcal{C}$ satisfies* $\operatorname{Hom}(\mathcal{B}, X) = 0$ *if and only if* $\operatorname{Hom}(\mathfrak{I}, X) = 0$.

Let us mention an immediate consequence: The smashing subcategories of $\mathcal{C}$ form a set of cardinality at most 2^κ where κ denotes the cardinality of the set of isomorphism classes of maps between compact objects in $\mathcal{C}$. For example, the stable homotopy category has precisely $2^{\aleph_0}$ smashing subcategories because, in this case $\kappa = \aleph_0$, and arithmetic localization gives rise to a smashing subcategory for every set of primes.

Given any class $\mathfrak{I}$ of maps in $\mathcal{C}$, we say that a localizing subcategory $\mathcal{B}$ is *generated* by $\mathfrak{I}$ if $\mathcal{B}$ is the smallest localizing subcategory of $\mathcal{C}$ such that every map in $\mathfrak{I}$ factors through some object in $\mathcal{B}$. For example, $\mathcal{B}$ is generated by a class $\mathfrak{I} = \{\mathrm{id}_{X_i} \mid i \in I\}$ of identity maps if and only if $\mathcal{B}$ is the smallest localizing subcategory containing X_i for all $i \in I$.

Corollary. *Every smashing subcategory is generated by a set of maps between compact objects.*

The statement of the corollary is a modified version of the following "telescope conjecture" which is based on conjectures of Ravenel [23, 1.33] and Bousfield [6, 3.4] for the stable homotopy category:

Every smashing subcategory is generated by a set of identity maps between compact objects.

In this generality, the conjecture is known to be false. In fact, Keller gives an example of a smashing subcategory which contains no non-zero compact object [14]. Despite some efforts of Ravenel [24], the conjecture remains open for the stable homotopy category.

The characterization of smashing subcategories leads to a classification in terms of certain ideals which we now explain. We denote by $\mathcal{C}_0$ the full triangulated subcategory of compact objects in $\mathcal{C}$ and call an ideal $\mathfrak{I}$ of maps in $\mathcal{C}_0$ *exact* if there exists an exact functor $f\colon \mathcal{C}_0 \to \mathcal{D}$ into a triangulated category $\mathcal{D}$ such that $\mathfrak{I} = \{\phi \in \mathcal{C}_0 \mid f(\phi) = 0\}$.

Theorem B. *Let $\mathcal{C}$ be a compactly generated triangulated category and suppose that every cohomological functor $\mathcal{C}_0^{\mathrm{op}} \to \mathrm{Ab}$ is isomorphic to $\operatorname{Hom}(-, X)|_{\mathcal{C}_0}$ for some object X in $\mathcal{C}$. Then the maps*

$$\mathcal{B} \mapsto \{\phi \in \mathcal{C}_0 \mid \phi \text{ factors through an object in } \mathcal{B}\} \quad \text{and}$$

$$\mathfrak{I} \mapsto \{X \in \mathcal{C} \mid \text{every map } C \to X,\ C \in \mathcal{C}_0, \text{ factors through a map } C \to D \text{ in } \mathfrak{I}\}$$

induce mutually inverse bijections between the set of smashing subcategories of $\mathcal{C}$ and the set of exact ideals in $\mathcal{C}_0$.

Note that the additional assumption on $\mathcal{C}$ in the preceding theorem is automatically satisfied if there are at most countably many isomorphism classes of maps between compact objects in $\mathcal{C}$; in particular the stable homotopy category has this property [21]. The classification of smashing subcategories has the following consequence.

Corollary. *A localizing subcategory $\mathcal{B}$ of $\mathcal{C}$ is smashing if and only if $\mathcal{B}$ is generated by a class of maps between compact objects in $\mathcal{C}$. Moreover, given any class $\mathfrak{I}$ of maps between compact objects in $\mathcal{C}$, there exists a localizing subcategory of $\mathcal{C}$ which is generated by $\mathfrak{I}$.*