Normal forms for representations of representation-finite algebras

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1 Introduction and main result

Why is representation theory of algebras important for physics? Besides well-known answers like the occurrence of symmetries, operator algebras or quantum groups in physical theories there is a less popular and more naive answer to this question. Namely, representation theory of finite-dimensional algebras deals with normal forms of matrices. But finding normal forms for matrices is an important technique all over physics. Probably a student encounters this for the first time when trying to solve a system of linear differential equations describing some harmonic oscillator or when trying to find the main axes of a solid body by diagonalising the inertia tensor. Representation theory of finite-dimensional algebras provides a general environment to deal with problems of this shape. The aim of this paper is to show how computer algebra is used to obtain information about a certain type of normal forms and to indicate how this could be made available for practical use.

Let us explain the way how representation theory of finite dimensional algebras deals with normal forms of matrices. For this purpose we need concepts like quivers and their representations introduced in [Ga1]. Recall that a quiver Q is nothing but a directed graph with a set of vertices Q_0 and a set of arrows Q_1 such that each arrow α in Q_1 has a unique initial point $s(\alpha)$ and a unique final point $t(\alpha)$ in Q_0. For a given field k the path algebra kQ is the vector space having as basis the paths in the quiver Q. A path of positive length p in Q is a sequence $\alpha_p \dots \alpha_1$ of arrows such that $s(\alpha_i) = t(\alpha_{i-1})$ for all $i = 2, \dots, p$. Moreover, to each vertex i there is attached a path of length 0 starting and ending in i. The path algebra kQ is endowed with the multiplication which is defined for two paths v and u as the concatenation of the two paths if the final point of u coincides with the initial point of v and 0 otherwise. Note, that kQ is not finite-dimensional if Q has an oriented cycle. Representations of a path algebra kQ are easy to describe as representations of Q (over the given field k). A representation X of Q is given by a vector spaces $X(i)$ for each vertex i in Q_0 and a linear map $X(\alpha) : X(s(\alpha)) \to X(t(\alpha))$ for each arrow α in Q_1.

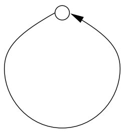
A homomorphism from a representation X to a representation Y of a quiver Q is a family $\phi = (\phi_i)_{i \in Q_0}$ of linear maps $\phi_i : X(i) \to Y(i)$ such that $Y(\alpha)\phi_{s(\alpha)} = \phi_{t(\alpha)}X(\alpha)$ for all arrows α. Such a homomorphism is called isomorphism if all the maps ϕ_i are bijective.

Two representations X, Y are said to be isomorphic if there exists an isomorphism from X to Y.

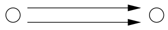
A representation X of Q is called finite-dimensional if all vector spaces $X(i)$ are finite-dimensional say of dimension $n(i)$. By choosing bases, each space $X(i)$ can be identified with $k^{n(i)}$. This forces each map $X(\alpha)$ to be a (left) multiplication by a $n(t(\alpha)) \times n(s(\alpha))$-matrix over k which we denote by $X(\alpha)$ as well. A representative of an isomorphism class of finite-dimensional representations having this shape is called a normal form.

The direct sum of a collection $X_1, \ldots , X_m$ of representations of a quiver Q is defined in the obvious way by building the direct sum of the vector spaces $X_1(i), \ldots , X_m(i)$ for each vertex i in Q_0. A representation X of Q is said to be indecomposable if it is not isomorphic to the direct sum of two non-zero representations. By the theorem of Krull-Remak-Schmidt any finite-dimensional representation is isomorphic to a direct sum of finitely many indecomposable representations which are unique up to permutation and isomorphism. Thus it is sufficient to find normal forms for all isomorphism classes of indecomposable representations.

The most popular normal forms are the Jordan block matrices which are the normal forms for the indecomposable representations of the one loop quiver.



Another classical example are the normal forms for matrix pencils (see [Gm]) which are the normal forms for the indecomposable representations of the Kronecker quiver below. For an application in perturbation theory we refer to [AF].



To what extent are quivers and their path algebras typical in the representation theory of algebras? If we wish to study a finite-dimensional representation X of a given k-algebra A, then we clearly can consider X as representation of the factor algebra of A by the annihilator ideal of X. Since this factor algebra is finite-dimensional, it suffices to consider finite-dimensional algebras when looking for normal forms for finite-dimensional indecomposable representations. It was emphasised by Gabriel (see [Ga1] and [Ga3]) that any finite-dimensional algebra over an algebraically closed field k (like $\mathbb{C}$) can be considered (up to Morita equivalence) as a factor algebra kQ/I of the path algebra kQ associated with a finite quiver Q. The ideal I of kQ can be chosen to be admissible which means that it is contained in the square of the ideal of kQ generated by the arrows but contains some power of this ideal. Note, that such an ideal can always be generated by finitely many generators $\rho_1, \ldots , \rho_r$ such that each of these 'relations' ρ_l is a linear combination $\rho_l = \sum_{q=1}^{m} \lambda_q u_q$ where $\lambda_q \in k$ and all the u_q are paths from a vertex i to a vertex j of Q_0. We observed above that representations of a quiver algebra kQ can be identified with the representations of Q over k. This transfers easily to representations of kQ/I. Namely, the representations of this factor algebra are just those representations X of Q satisfying all the relations ρ_l meaning that $0 = \sum_{q=1}^{m} \lambda_q X(u_q)$. Here we use the abbreviation $X(u) := X(\alpha_p) \cdots X(\alpha_1)$ when considering a path $u = \alpha_p \ldots \alpha_1$ in Q. Thus