# Cyclotomic Schur algebras
# and blocks of cyclic defect

by

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**Abstract**

An explicit classification is given of blocks of cyclic defect of cyclotomic Schur algebras and of cyclotomic Hecke algebras, over discrete valuation rings.

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## 1 Introduction

A fundamental result of representation theory of finite groups is the classification of blocks of cyclic defect. The aim of this note is to use this classification as a tool for classifying blocks of cyclotomic Hecke algebras and of the associated Schur algebras, over discrete valuation rings. This reproves and generalizes results of Xi and of Erdmann for classical Schur algebras over fields.

We define two classes of $R$–orders which will be the outcome of the classification. Here $R$ is a discrete valuation ring with maximal ideal $\pi$, field of fractions $K$ and residue field $k = R/\pi$. The notation $R - R$ (two tied copies of $R$) means the set $\{(a, b) \in R \times R : a - b \in \pi\}$.

Fix a natural number $n \in \mathbb{N}$. The $R$–order $\Lambda_n$ is the following subset of $n - 1$ copies of $2 \times 2$–matrices over $K$ and one copy of $1 \times 1$–matrices over $K$ (with ties between the first and the second copy, the second and the third copy and so on):

$$\begin{pmatrix} R & R \\ \pi & R \end{pmatrix} \diagup \begin{pmatrix} R & R \\ \pi & R \end{pmatrix} \diagup \begin{pmatrix} R & R \\ \pi & R \end{pmatrix} \diagup \cdots \diagup \begin{pmatrix} R & R \\ \pi & R \end{pmatrix} \diagup \begin{pmatrix} R \end{pmatrix}$$

The $R$–order $\Gamma_n$ is the following subset of $n-2$ copies of $2\times 2$–matrices over $K$ and two copies of $1\times 1$–matrices over $K$ (with ties between the first and the second copy, the second and the third copy and so on):

$$\left( R \right) \begin{pmatrix} R & R \\ \pi & R \end{pmatrix} \begin{pmatrix} R & R \\ \pi & R \end{pmatrix} \cdots \begin{pmatrix} R & R \\ \pi & R \end{pmatrix} \left( R \right)$$

For finite groups and also for Hecke algebras, there is a well–defined notion of blocks of cyclic defect. By Schur–Weyl duality, the blocks of a Hecke algebra and of the associated Schur algebra correspond to each other. Thus it makes sense to talk about blocks of cyclic defect of cyclotomic Schur algebras as well (see section two for definitions and references).

**Theorem 1.1** *Fix $R$ as above.*

*(a) Let $\Lambda$ be an $R$–order which is a block of cyclic defect of a cyclotomic Schur algebra. Then $\Lambda$ is Morita equivalent to $\Lambda_n$ for some $n$.*

*(b) Let $\Gamma$ be an $R$–order which is a block of cyclic defect of a cyclotomic Hecke algebra. Then $\Gamma$ is Morita equivalent to $\Gamma_n$ for some $n$.*

The proof of the theorem will be based on proposition 3.1, which gives the same classification in an abstract setup containing the situation of the theorem.

We note that the integer $n$ can be read off from the group theoretic data (see [1], section 17).

It is an important feature of cyclotomic Hecke and Schur algebras that they generalize both type $A$ and type $B$ situations. In the case of cyclic defect, the theorem however tells us that type $B$ does not contribute any new examples.

**Corollary 1.2** *The order $\Lambda_n$ has infinite global dimension. The order $\Gamma_n$ has global dimension $2n-1$.*

*For each $n$, the order $\Gamma_n$ contains an idempotent $e$ such that $e\Gamma_n e$ is isomorphic to $\Lambda_n$. Moreover, the left $\Gamma_n$–lattice $\Gamma_n e$ is a full tilting module of the quasi–hereditary order $\Gamma_n$, and the $\Gamma_n$–$\Lambda_n$–bimodule $\Gamma_n e$ induces a Schur–Weyl duality*

$$\Lambda_n \simeq End_{\Gamma_n}(\Gamma_n e),$$

$$\Gamma_n \simeq End_{\Lambda_n}(\Gamma_n e)$$