DECOMPOSING THICK SUBCATEGORIES OF THE STABLE MODULE CATEGORY

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ABSTRACT. Let $\underline{\mathrm{mod}}\, kG$ be the stable category of finitely generated modular representations of a finite group G over a field k. We prove a Krull-Schmidt theorem for thick subcategories of $\underline{\mathrm{mod}}\, kG$. It is shown that every thick tensor-ideal $\mathcal{C}$ of $\underline{\mathrm{mod}}\, kG$ (i.e. a thick subcategory which is a tensor ideal) has a (usually infinite) unique decomposition $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$ into indecomposable thick tensor-ideals. This decomposition follows from a decomposition of the corresponding idempotent kG-module $E_\mathcal{C}$ into indecomposable modules. If $\mathcal{C} = \mathcal{C}_W$ is the thick tensor-ideal corresponding to a closed homogeneous subvariety W of the maximal ideal spectrum of the cohomology ring $H^*(G, k)$, then the decomposition of $\mathcal{C}$ reflects the decomposition $W = \bigcup_{i=1}^n W_i$ of W into connected components.

INTRODUCTION

In modular representation theory of finite groups, one frequently passes to the stable module category which is a triangulated category. Following ideas from stable homotopy theory, Benson, Carlson, and Rickard studied in a number of papers the lattice of thick subcategories of the stable module category. In [4], these authors obtain for p-groups a description of this lattice which is an analogue of a result for the stable homotopy category due to Devinatz, Hopkins, and Smith [8, 11]. In this paper we prove a Krull-Schmidt theorem for this lattice which we now explain.

Let $\Lambda = kG$ be the group algebra of a finite group G over a field k. The stable category $\underline{\mathrm{mod}}\,\Lambda$ of finitely generated Λ-modules is a triangulated category and carries a tensor product which is induced from the tensor product on $\mathrm{mod}\,\Lambda$. A full subcategory $\mathcal{C}$ of $\underline{\mathrm{mod}}\,\Lambda$ is called a *thick tensor-ideal* in $\underline{\mathrm{mod}}\,\Lambda$ if $\mathcal{C}$ is a thick subcategory and $X \otimes Y \in \mathcal{C}$ for all $X \in \mathcal{C}$ and $Y \in \underline{\mathrm{mod}}\,\Lambda$. Given a thick tensor-ideal $\mathcal{C}$, we say that a family $(\mathcal{C}_i)_{i\in I}$ of thick tensor-ideals in $\underline{\mathrm{mod}}\,\Lambda$ is a *decomposition* of $\mathcal{C}$ if

(1) the objects in $\mathcal{C}$ are the finite coproducts of objects from the $\mathcal{C}_i$;
(2) $\mathcal{C}_i \cap \mathcal{C}_j = 0$ for all $i \neq j$ in I.

A decomposition $(\mathcal{C}_i)_{i\in I}$ of $\mathcal{C}$ is denoted by $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$, and we say that $\mathcal{C}$ is *indecomposable* if $\mathcal{C} \neq 0$ and any decomposition $\mathcal{C} = \mathcal{C}_1 \coprod \mathcal{C}_2$ implies $\mathcal{C}_1 = 0$ or $\mathcal{C}_2 = 0$.

Theorem A. *Every thick tensor-ideal $\mathcal{C}$ in $\underline{\mathrm{mod}}\,\Lambda$ has a unique decomposition $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$ into indecomposable thick tensor-ideals. Conversely, given any family $(\mathcal{C}_i)_{i\in I}$ of thick tensor-ideals in $\underline{\mathrm{mod}}\,\Lambda$ satisfying $\mathcal{C}_i \cap \mathcal{C}_j = 0$ for all $i \neq j$ in I, there exists a thick tensor-ideal $\mathcal{C}$ such that $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$.*

Before we discuss this result in more detail let us mention that there seems to be no analogous statement for the stable homotopy category. Our analysis of thick subcategories in $\underline{\mathrm{mod}}\,\Lambda$ is based on the analysis of certain objects in the stable category $\underline{\mathrm{Mod}}\,\Lambda$ of all Λ-modules. In fact, we are able to compute the centre $Z[\mathcal{C}^{\amalg}]$ of the localizing subcategory $\mathcal{C}^{\amalg}$ of $\underline{\mathrm{Mod}}\,\Lambda$ which is generated by a thick tensor-ideal $\mathcal{C}$ in $\underline{\mathrm{mod}}\,\Lambda$. Recall that the centre of an additive category is the endomorphism ring of the identity functor.

We prove that $Z[\mathcal{C}^{\amalg}]$ is a product of commutative local rings, and the corresponding decomposition $\mathrm{id}_{\mathcal{C}} = (e_i)_{i\in I}$ of the identity functor into central idempotents gives the decomposition $\mathcal{C} = \coprod_{i\in I}\mathcal{C}_i$ if we define $\mathcal{C}_i = e_i(\mathcal{C})$ for every i. Note that the centre of the stable homotopy category is $\mathbb{Z}$ which is not a local ring. Therefore one cannot expect an analogue of Theorem A for the stable homotopy category because any Krull-Schmidt theorem requires local endomorphism rings for the indecomposable objects.

In [18], Rickard introduced for every thick subcategory $\mathcal{C}$ of $\underline{\mathrm{mod}}\,\Lambda$ a particular Λ-module $E_{\mathcal{C}}$. It is obtained from the trivial representation k by applying a right adjoint $e\colon \underline{\mathrm{Mod}}\,\Lambda \to \mathcal{C}^{\amalg}$ of the inclusion $\mathcal{C}^{\amalg} \to \underline{\mathrm{Mod}}\,\Lambda$, i.e. $E_{\mathcal{C}} = e(k)$. If $\mathcal{C}$ is a tensor-ideal, then the canonical map $E_{\mathcal{C}} \to k$ induces an isomorphism $E_{\mathcal{C}} \otimes E_{\mathcal{C}} \to E_{\mathcal{C}}$ in $\underline{\mathrm{Mod}}\,\Lambda$, and therefore $E_{\mathcal{C}}$ is called *idempotent*. In recent years, the study of such modules has become one of the most successful methodologies in modular representation theory of finite groups. We obtain Theorem A from the following result since $Z[\mathcal{C}^{\amalg}] \simeq \underline{\mathrm{End}}_{\Lambda}(E_{\mathcal{C}})$.

Theorem B. *The Λ-module $E_{\mathcal{C}}$ has, up to isomorphism, a unique decomposition $E_{\mathcal{C}} = \coprod_{i\in I} E_i$ into indecomposable Λ-modules with local endomorphism ring. Moreover, the endomorphism ring of $E_{\mathcal{C}}$ has the following properties:*

(1) $\bigcap_{n\in\mathbb{N}} \mathfrak{r}^n = 0$ *for the Jacobson radical $\mathfrak{r}$ of $\underline{\mathrm{End}}_{\Lambda}(E_{\mathcal{C}})$;*
(2) *if $\mathcal{C}$ is a tensor-ideal, then $\underline{\mathrm{End}}_{\Lambda}(E_{\mathcal{C}}) \simeq \prod_{i\in I} \underline{\mathrm{End}}_{\Lambda}(E_i)$*

The decomposition of the module $E_{\mathcal{C}}$ is the consequence of a more general result. In fact, we introduce the concept of an endofinite object in a compactly generated triangulated category and prove that any endofinite object decomposes uniquely into indecomposable objects. Theorem B then follows since $E_{\mathcal{C}}$ is an endofinite object in the localizing subcategory of $\underline{\mathrm{Mod}}\,\Lambda$ which is generated by $\mathcal{C}$.

Interesting examples of endofinite objects also arise in stable homotopy theory. For instance, the classifying space BG of G is endofinite in the category of spectra [15]. Therefore we obtain a new proof for the existence of a stable splitting of BG which does not involve Segal's conjecture; see Benson's survey [2] for more details.

Let us end this introduction with a geometric interpretation of Theorem A in terms of the projective prime ideal spectrum of the cohomology ring $H^*(G,k)$. More precisely, we denote by $\mathcal{V}_G(k)$ the set of closed homogeneous irreducible non-zero subvarieties of the maximal ideal spectrum $V_G(k)$ of $H^*(G,k)$. Generalizing the definition of a closed homogeneous subvariety of $V_G(k)$, Benson, Carlson, and Rickard consider in [3] subsets $\mathcal{W}$ of $\mathcal{V}_G(k)$ which are closed under *specialization*, i.e. $W \in \mathcal{W}$ and $V \subseteq W$ implies $V \in \mathcal{W}$. For instance, any closed homogeneous subvariety W of $V_G(k)$ is determined by $\mathcal{W} = \{V \in \mathcal{V}_G(k) \mid V \subseteq W\}$. Given any subset $\mathcal{W}$ of $\mathcal{V}_G(k)$ closed under specialization, we say that $\mathcal{W}$ is *projectively connected* if any decomposition $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ into subsets of $\mathcal{V}_G(k)$ closed under specialization such that $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$ implies $\mathcal{W} = \mathcal{W}_1$ or $\mathcal{W} = \mathcal{W}_2$. The following result generalizes the decomposition $W = \bigcup_{i=1}^n W_i$ of a closed homogeneous subvariety of $V_G(k)$ into connected components.

Theorem C. *Every subset $\mathcal{W}$ of $\mathcal{V}_G(k)$ closed under specialization has a unique decomposition $\mathcal{W} = \bigcup_{i\in I} \mathcal{W}_i$ into projectively connected non-empty subsets of $\mathcal{V}_G(k)$ closed under specialization such that $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$ for all $i \neq j$ in I.*

1. Endofiniteness

Let $\mathcal{T}$ be a triangulated category [19] and suppose that arbitrary coproducts exist in $\mathcal{T}$. An object X in $\mathcal{T}$ is called *compact* if for every family $(Y_i)_{i\in I}$ in $\mathcal{T}$ the canonical map $\coprod_i \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \coprod_i Y_i)$ is an isomorphism. We denote by $\mathcal{T}_0$ the full