A combinatorial algorithm for the minimum $(2, r)$-metric problem and some generalizations

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Abstract. Let $G = (V, E)$ be a graph with nonnegative integer capacities $c(e)$ of the edges $e \in E$, and let μ be a metric that establishes distances on the pairs of elements of a subset $T \subseteq V$. In the *minimum 0-extension problem* $(*)$, one is asked for finding a metric m on V such that m coincides with μ within T, each $x \in V$ is at zero distance from some $t \in T$, and the value $\sum(c(e)m(e) : e \in E)$ is as small as possible. When $T = \{s, t\}$ and $\mu(s, t) = 1$, this turns into the classical minimum (undirected) cut problem. When μ is the path metric of the complete bipartite graph $K_{2,r}$, $(*)$ is specified to be the *minimum* $(2, r)$*-metric problem*. It is known that such a problem can be solved in strongly polynomial time by use of the ellipsoid method.

We develop a polynomial time algorithm for the minimum $(2, r)$-metric problem, using only "purely combinatorial" means. The algorithm simultaneously solves a certain associated integer multiflow problem. We then apply this algorithm to solve $(*)$ for a wider class of metrics μ, give other results and raise open questions.

Key words: finite metric, cut, metric extension, multicommodity flow.

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1. Introduction

By a *metric* on a set S we mean a function $d : S \times S \to \mathbb{R}_+$ that establishes *distances* on the pairs of elements of S satisfying (i) $d(x,x) = 0$, (ii) $d(x,y) = d(y,x)$, and (iii) $d(x,y) + d(y,z) \geq d(x,z)$, for all $x,y,z \in S$. We usually write $d(xy)$ in place of $d(x,y)$ and allow $d(xy) = 0$ for some $x \neq y$. A special case of metrics is the *path metric* d^Γ of a connected graph $\Gamma = (S,W)$, i.e., $\mathrm{d}^\Gamma(xy)$ is the minimum number of edges of a path in Γ connecting x and y.

We consider an undirected graph $G = (V,E)$ whose edges $e \in E$ have nonnegative integer *capacities* $c(e)$, and a subset $T \subseteq V$ of nodes called *terminals*. Let μ be a metric on T. A metric m on V is said to be an *extension* of μ to V if the submetric of m on T is just μ (i.e., m and μ coincide within T), and to be a *0-extension* if, in addition, for each $x \in V$, there is $t \in T$ such that $m(tx) = 0$. If μ is a *positive* metric, i.e., $\mu(st) > 0$ for any $s \neq t$, then each 0-extension m of μ to V one-to-one corresponds to a *T-partition*, i.e., a partition of V into $|T|$ subsets $\{X_t : t \in T\}$ where each X_t contains exactly one terminal, namely, t. This correspondence is defined by setting $X_t = \{x \in V : m(tx) = 0\}$.

A path in G whose ends are different elements of T is called a *T-path*. A (c-admissible) *multiflow* f consists of T-paths $P_1, \ldots, P_k$ in G along with nonnegative real weights $\lambda_1 = \lambda(P_1), \ldots, \lambda_k = \lambda(P_k)$ satisfying the capacity constraint

$$f^e := \sum(\lambda_i : P_i \text{ contains } e) \leq c(e) \quad \text{for all } e \in E.$$

Define $\langle \mu, f \rangle$, the *μ-value* of f, to be $\sum(\mu(u_iv_i)\lambda_i : i = 1, \ldots, k,\ u_i$ and v_i are the ends of P_i). If all λ_i's are integers, f is called an *integer* multiflow. Consider the following four problems:

(1.1) Find a 0-extension m of μ to V with $c \cdot m := \sum(c(e)m(e) : e \in E)$ as small as possible;

(1.2) Find an extension m of μ to V with $c \cdot m$ as small as possible;

(1.3) Find an integer multiflow f whose μ-value is as large as possible;

(1.4) Find a multiflow f whose μ-value is as large as possible.

(Problem (1.1) is also known as the multifacility location problem, cf. [TFL].) Let $\tau = \tau(G,c,\mu)$ and $\tau^* = \tau^*(G,c,\mu)$ denote the minimum $c \cdot m$ in (1.1) and (1.2), respectively, and let $\nu = \nu(G,c,\mu)$ and $\nu^* = \nu^*(G,c,\mu)$ denote the maximum μ-values in (1.3) and (1.4), respectively. For given G, c, μ, an extension m with $c \cdot m = \tau^*$ is called *minimum* and a multiflow f with $\langle \mu, f \rangle = \nu^*$ is called *maximum*, and similarly for 0-extensions and integer multiflows. Since (1.1) is a strengthening of (1.2), and (1.3)