Iterated Monoidal Categories II

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Introduction

In recent years many examples of a new kind of algebraic structure on a category have been discovered: braided monoidal categories, such as categories of representations of quantum groups (cf. [11] & [13]). It is striking to note that there appears to be a connection between braided monoidal categories and 2-fold loop spaces similar to the one between monoidal categories and 1-fold loop spaces or between symmetric monoidal categories and infinte loop spaces. It is shown in [9] that the group completion of the nerve of a braided monoidal category is a 2-fold loop space. This result raises an obvious question: what algebraic structure on a category corresponds to an n-fold loop structure for $3 \leq n < \infty$? Unfortunately the proof of [9] sheds no light on this matter.

In [10] we proposed a solution to this problem based on pursuing an analogy to the tautology that an n-fold loop space is a loop space in the category of $(n-1)$-fold loop spaces. Noting the correspondence between loop spaces and monoidal categories, we iteratively defined the notion of n-fold monoidal category as a monoid in the category of $(n-1)$-fold monoidal categories. There are some subtleties involved in making this definition work: one has to define "monoidal" up to a requisite degree of what category theorists call "laxness". We showed that the group completion of the nerve of an n-fold monoidal category is an n-fold loop space. But [10] raised more questions than it answered: what is the precise relation before group completion of an n-fold monoidal category to an algebra over the little n-cubes operad, which as shown in [4] and [15] characterizes the notion of n-fold loop space? A priori, it is conceivable that after group completion the nerve of an n-fold monoidal category has more structure, it might even be an infinite loop space. E.g. the version of n-fold monoidal category investigated by Joyal and Street [11] gives an infinite loop space for $n \geq 3$.

The present paper answers this question. Our main result is that our notion of iterated monoidal category precisely corresponds to the notion of an algebra over the little n-cubes operad for all n. Firstly the group completion of the nerve of such a category is an n-fold loop space. Secondly one can form an operad in the category of small categories which parametrizes the algebraic structure of an n-fold monoidal category. We show that the nerve of this categorical operad is a topological operad which is equivalent, as an operad, to the little n-cubes operad. Thus our result can be regarded as an algebraic characterization of the notion of n-fold loop space. We also note that this algebraically defined operad is a finite simplicial operad and is closely related to the Milgram construction [18] for $\Omega^n S^n X$.

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1 n-fold Monoidal Categories

In this section we gradually develop the notion of iterated monoidal category. We start by recalling the standard notion of monoidal category and defining a slightly nonstandard variant of the notion of monoidal functor.

Definition 1.1. A *(strict) monoidal category* is a category $\mathcal{C}$ together with a functor $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object 0 such that

1. $\Box$ is strictly associative.
2. 0 is a strict 2-sided unit for $\Box$.

A monoidal functor $(F,\eta): \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor F such that $F(0)=0$ together with a natural transformation

$$\eta_{A,B}: F(A)\Box F(B) \to F(A\Box B),$$

which satisfies the following conditions

1. Internal Associativity: The following diagram commutes

$$\begin{tikzcd} F(A)\Box F(B)\Box F(C) \arrow[r, "\eta_{A,B}\Box id_{F(C)}"] \arrow[d, "id_{F(A)}\Box\eta_{B,C}"] & F(A\Box B)\Box F(C) \arrow[d, "\eta_{A\Box B,C}"] \\ F(A)\Box F(B\Box C) \arrow[r, "\eta_{A,B\Box C}"] & F(A\Box B\Box C) \end{tikzcd}$$

2. Internal Unit Conditions: $\eta_{A,0} = \eta_{0,A} = id_{F(A)}$.

Given two monoidal functors $(F,\eta): \mathcal{C} \to \mathcal{D}$ and $(G,\zeta): \mathcal{D} \to \mathcal{E}$, we define their composite to be the monoidal functor $(GF,\xi): \mathcal{C} \to \mathcal{E}$, where ξ denotes the composite

$$GF(A)\Box GF(B) \xrightarrow{\zeta_{F(A),F(B)}} G\big(F(A)\Box F(B)\big) \xrightarrow{G(\eta_{A,B})} GF(A\Box B).$$

(It is an exercise to check that ξ satisfies the associativity condition above.) We denote by **MonCat** the category of monoidal categories and monoidal functors. Note that the usual product in **Cat** defines a product in **MonCat**.

Remark 1.2. It is usually required in standard definitions of the notion of monoidal functor that η be an isomorphism. As we will discuss below, it is crucial for us not to make this requirement.

Definition 1.3. A *2-fold monoidal category* is a monoid in **MonCat**. This means that we are given a monoidal category $(\mathcal{C}, \Box_1, 0)$ and a monoidal functor $(\Box_2, \eta): \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ which satisfies

1. External Associativity: the following diagram commutes in **MonCat**

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "(\Box_2,\eta)\times id_{\mathcal{C}}"] \arrow[d, "id_{\mathcal{C}}\times(\Box_2,\eta)"] & \mathcal{C}\times\mathcal{C} \arrow[d, "(\Box_2,\eta)"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "(\Box_2,\eta)"] & \mathcal{C} \end{tikzcd}$$

2. External Unit Conditions: the following diagram commutes in **MonCat**

$$\begin{tikzcd} \mathcal{C}\times 0 \arrow[r, "\subseteq"] \arrow[d, "\cong"] & \mathcal{C}\times\mathcal{C} \arrow[d, "(\Box_2,\eta)"] & 0\times\mathcal{C} \arrow[l, "\supseteq"'] \arrow[d, "\cong"] \\ \mathcal{C} \arrow[r, "="] & \mathcal{C} & \mathcal{C} \arrow[l, "="'] \end{tikzcd}$$

Explicitly this means that we are given a second associative binary operation $\Box_2: \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, for which 0 is also a two-sided unit. Moreover we are given a natural transformation

$$\eta_{A,B,C,D}: (A\Box_2 B)\Box_1(C\Box_2 D) \to (A\Box_1 C)\Box_2(B\Box_1 D).$$