ON THE DISTRIBUTION OF AR-COMPONENTS OF RESTRICTED LIE ALGEBRAS

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Abstract

In this paper we study the distribution of indecomposable modules of the reduced enveloping algebra $u(L,\chi)$ associated to a finite dimensional restricted Lie algebra $(L,[p])$. Each component of the stable Auslander-Reiten quiver possesses only finitely modules of a given dimension. We combine this fact with geometric techniques in order to produce families of components of type $\mathbb{Z}[A_\infty]$.

0. Introduction and Preliminaries

Let F be an algebraically closed field, Λ a finite dimensional F-algebra of infinite representation type. If Λ is tame, it follows from Brauer Thrall II and the work of Crawley-Boevey [5, Corollary E] that there exists a number $d \in \mathbb{N}$, such that for each multiple ℓd there are infinitely many non-isomorphic indecomposable Λ-modules of dimension ℓd. In case Λ is wild, this conclusion continues to hold (with d even) since there is a representation embedding from the module category of the Kronecker algebra $F[X,Y]/(X^2,Y^2)$ into the module category of Λ. A related question concerning the distribution of the dimensions within a given Auslander-Reiten component was raised by Ringel [30]: Given an AR-component Θ and $d > 0$, is the set

$$\Theta(d) := \{[M] \in \Theta \ ; \ dim_F M \leq d\}$$

finite? While the answer is affirmative for some classes of algebras (cf. for instance [5, Corollary F]), there exist symmetric algebras which admit infinite $\Theta(d)$ (cf. [24]).

In this note we shall study questions related to the distribution of AR-components for reduced enveloping algebras of restricted Lie algebras. Our results, some of which will be

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applied in a forthcoming paper [12], illustrate the utility of the geometric methods that are based on support varieties.

In the sequel we shall consider finite dimensional restricted Lie algebras $(L, [p])$, defined over an algebraically closed field F of characteristic $p > 0$. The universal enveloping algebra of L will be denoted $\mathcal{U}(L)$. Given a linear form $\chi \in L^*$, the *χ-reduced enveloping algebra* of L is defined by

$$u(L, \chi) := \mathcal{U}(L)/(\{x^p - x^{[p]} - \chi(x)^p 1 \;\; ; \;\; x \in L\}).$$

We refer the interested reader to [32] concerning basic facts of restricted Lie algebras and their enveloping algebras. A few properties of $u(L, \chi)$ that are relevant for our purposes are summarized in the following list (cf. [6, 8, 13]).

- The algebra $u(L, \chi)$ is finite dimensional, more precisely $dim_F u(L, \chi) = p^{dim_F L}$.
- $u(L, \chi)$ is a Frobenius algebra with Nakayama automorphism μ uniquely determined by
$$\mu(x) = x - tr(ad\, x)1 \quad \forall\, x \in L.$$
- The restricted enveloping algebra $u(L) := u(L, 0)$ is a Hopf algebra.
- Let Θ be a component of the stable Auslander-Reiten quiver $\Gamma_s(L, \chi)$ of $u(L, \chi)$. Then Θ is either a finite or infinite tube of rank $k \in \{1, p\}$, or
$$\Theta \cong \mathbb{Z}[A_\infty],\ \mathbb{Z}[A_\infty^\infty],\ \mathbb{Z}[D_\infty],\ \mathbb{Z}[\tilde{A}_n],\ \mathbb{Z}[\tilde{A}_{12}],\ \mathbb{Z}[\tilde{D}_n],\ \mathbb{Z}[\tilde{E}_\ell] \quad (6 \leq \ell \leq 8).$$

In the following all $u(L, \chi)$-modules are assumed to be finite dimensional. We also set $\Gamma_s(L) := \Gamma_s(L, 0)$.

Let M be a $u(L, \chi)$-module. In view of the work of Friedlander and Parshall (cf. [15, 16, 17]) and Jantzen [22] we define the *cohomological support variety* of M via

$$\mathcal{V}_L(M) := \{x \in L \;;\; x^{[p]} = 0 \; and \; M|_{\langle x \rangle} \; is \; not \; free\}.$$

Here $M|_{\langle x \rangle}$ denotes the restriction of M to the subalgebra of $u(L, \chi)$ that is generated by x.

It is well-known (cf. [15, 16, 17]) that $\mathcal{V}_L(M)$ is a conical variety whose dimension coincides with the complexity $c_{u(L,\chi)}(M)$. In particular, M is projective if and only if $\mathcal{V}_L(M) = \{0\}$. Moreover, if $\Theta \subset \Gamma_s(L, \chi)$ is a component, then we have $\mathcal{V}_L(M) = \mathcal{V}_L(N) \quad \forall\, [M], [N] \in \Theta$. Accordingly, we may define

$$\mathcal{V}_L(\Theta) := \mathcal{V}_L(M) \qquad [M] \in \Theta.$$

After providing an affirmative answer to Ringel's question (Theorem 1.2), we turn in §2 to algebraic Lie algebras, i.e., those where $L = Lie(G)$ is the Lie algebra of an algebraic group G. In this setting the stabilizer G_χ of χ (with respect to the coadjoint action) operates on $\Gamma_s(L, \chi)$ via automorphisms. If G_χ is reductive (e.g., if G is reductive and χ is semisimple), then the components invariant relative to this action are shown to consist of modules that are stable under the twist by the adjoint representation. In the same vein, we employ