The Storage Capacity of Generalized Hopfield Models with Semantically Correlated Patterns

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Abstract

We analyze the storage capacity of a variant of the Hopfield model with semantically correlated patterns ξ_i^ν (that is the patterns ξ^ν themselves are correlated but consist of independent components ξ_i^ν). We show that there is a class of generalized Hopfield models of neural networks with N neurons that can store $\frac{N}{\gamma \log N}$ or αN spatially correlated patterns (depending on which notion of storage is used), provided that the correlation is generated by a homogeneous Markov chain. The quantities γ and α are independent of the correlations such that these genralized Hopfield models may be regarded as the legitimate representative of the standard Hopfield model in the presence of semantically correlated data.

1 Introduction

The so-called Hopfield model has been intensively studied by physicists, mathematicians and computer scientists since its introduction by Pastur and Figotin [FP77] in 1977 as a solvable model of a spin glass and its reinterpretation by Hopfield [Ho82] as a neural network, or more precisely as an auto-associative memory. In the last two decades the literature on the Hopfield model has become enormously wide and basically splits into the two branches mentioned above: the spin glass aspect of the Hopfield model – a field which is intrinsically related to the statistical mechanics of the model – and the neural networks aspect, which is connected to its (zero temperature) dynamics.

As we do not focus on the spin glass aspect here, we just refer the reader to the excellent survey paper by Bovier and Gayrard [BG98], and the literature cited there, for an overview of the mathematically rigorous results in this area as well as to the fundamental work by Talagrand [T98], which established a breakthrough especially concerning the validity of the so called replica symmetric solution.

Concerning the neural networks facet of the model, among others, the pioneering papers of McEliece, Posner, Rodemich, and Venkatesh [MPRV87] and Newman [N88]

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have to be mentioned as well as those by Bovier and Gayrard [BG92] and Loukianova [Lou94], [Lou97]. An overview over this field can be found in the books by Amit [A89], and Hertz, Krogh, and Palmer [HKP91], as well as in the survey article by Petritis [P96].

It also should be mentioned that besides the mathematically rigorous results there is a variety of physical heuristics about this model, the starting point of which has been the observation that the Hopfield model resembles the Sherrington-Kirkpatrick model of spin glasses made by Amit, Gutfreund and Sompolinsky [AGS87].

In order to describe the essence of our contribution we have to go a little more into detail and introduce some notations. As we will concentrate on the Hopfield model as an associative memory, that is a storage device that is able to reobtain stored information from some input data, we will begin with the information to be stored. This information is encoded in sequences of plus or minus ones of length N - the so called patterns – and we assume that we want to store $M = M(N)$ of such patterns. So altogether this gives an $M \times N$ matrix $((\xi_i^\mu)_{i=1\dots,N})_{\mu=1,\dots,M}$ of ± 1. The central question in this aspect of the Hopfield model is how many of these patterns we can actually store, that is how M scales with N. Up to now this is, of course, not a well posed problem, since we have not defined the Hopfield model yet, nor did we say, what storage means, and finally we have not specified, how the ξ_i^μ are chosen.

Given the matrix of the patterns $(\xi_i^\mu)_{i,\mu}$ the Hopfield model may either be associated with the function (the Hamiltonian of the system)

$$H_N(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^{N} \sum_{\mu=1}^{N} \sigma_i \sigma_j \xi_i^\mu \xi_j^\mu \qquad \sigma \in \{-1,+1\}^N \tag{1}$$

on $\{-1,+1\}^N$ or, equivalently with the corresponding Gibbs measure at inverse temperature $\frac{1}{\beta} > 0$ on $\{-1,+1\}^N$

$$\varrho_{N,\beta}(\sigma) := \frac{\exp(-\beta H_N(\sigma))}{\sum_{\sigma'} \exp(-\beta H_N(\sigma'))} \qquad \sigma \in \{-1,+1\}^N. \tag{2}$$

We will henceforth refer to the model described by equations (1) and (2) as the standard Hopfield model. Let us mention that H_N may be rewritten with the help of the overlap vector $(m_N^\mu(\sigma))_{\mu=1,\dots,M}$

$$m_N^\mu(\sigma) := \frac{1}{N} \sum_{j=1}^{N} \sigma_j \xi_j^\mu \tag{3}$$

in the following convenient way

$$H_N(\sigma) = -\frac{N}{2} ||m_N(\sigma)||^2$$

where $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^M$.

The idea behind this setup is that, if we assume for a moment orthogonal patterns, that is (ξ^μ) which fulfill

$$m_N^\mu(\xi^\nu) = \delta_{\mu,\nu},$$