Modules of Solvable Infinitesimal Groups and the Structure of Representation-Finite Cocommutative Hopf Algebras

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Abstract

Let k be an algebraically closed field k of characteristic $p \geq 3$. In this paper we extend the representation and block theory of solvable infinitesimal groups to determine the structure of those cocommutative Hopf algebras over k that have finite representation type.

0. Introduction

In continuation of earlier work [7], we study in this paper the structure of cocommutative Hopf algebras of finite representation type. Since every such algebra is the group algebra $H(\mathcal{G})$ of a finite algebraic group $\mathcal{G}$, this problem has to interrelated aspects. On the one hand, one has to understand the structure of the group scheme $\mathcal{G}$, while on the other the Morita equivalence class of $H(\mathcal{G})$ has to be determined. Although we shall here be mainly concerned with the latter, structural information on the underlying group schemes enters crucially at several points.

Experience shows that the theory naturally divides into two parts that tend to exhibit different features and require different methods. The classical theory deals with group algebras of constant groups. Here the representation theory of each block is entirely governed by the structure of its defect group. Our main focus are therefore the group algebras of infinitesimal groups, where support varieties have proven to be an effective tool. If $\mathcal{G}$ has height ≤ 1, methods related to this notion lead to a complete description of the structure of the representation-finite blocks of $H(\mathcal{G})$ (cf. [4]). If one passess to groups of arbitrary height, one still has enough information to pin down the structure of the blocks and the stable Auslander-Reiten quiver of the infinitesimal representation-finite groups. In fact, one

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basic feature from the theory of restricted Lie algebras retains its validity: the simple modules of such a block can be obtained from one representative by repeated twisting with the Nakayama automorphism ν of $H(\mathcal{G})$, with each block having either 1 or $ord(\nu)$ simple modules.

We begin by collecting notation and general conventions that will be in force throughout the paper. With regard to algebraic groups we will follow [3] with the exception of quotients. These will be denoted $\mathcal{G}/\mathcal{H}$ rather than $\mathcal{G}\tilde{/}\mathcal{H}$. For a Hopf algebra H we will employ the Heyneman-Sweedler sigma notation for the coproduct:

$$\Delta : H \longrightarrow H \otimes_k H \quad ; \quad h \mapsto \sum_{(h)} h_{(1)} \otimes h_{(2)}.$$

The counit and the antipode of H will be denoted by ε and η, respectively. Moreover, $H^\dagger := ker\,\varepsilon$ is the augmentation ideal of H. Aside from the regular representation, H also acts on itself via the (left) *adjoint representation* that is given by

$$h \cdot x := \sum_{(h)} h_{(1)} x \eta(h_{(2)}) \qquad \forall\, h, x \in H.$$

The corresponding H-module will occasionally be written as H_{ad}.

By an H-module, we will always understand a left H-module, and all k-vector spaces are assumed to be of finite dimension. Given an H-module M and a k-algebra automorphism $\alpha \in Aut_k(H)$ of H, we let $M^{(\alpha)}$ be the H-module with underlying k-space M and twisted action

$$h \cdot m := \alpha^{-1}(h)m \qquad \forall\, h \in H,\ m \in M.$$

Since $M \mapsto M^{(\alpha)}$ is an equivalence on the category mod_H of all H-modules, the group $Aut_k(H)$ operates on the set of isoclasses of H-modules such that the sets of indecomposables, simples, and projectives are stable. Note that the group of inner automorphisms fixes every isoclass of mod_H.

Let $G(H)$ be the group of group-like elements of H. Then $G(H^*)$, the set of group-likes of the dual Hopf algebra, coincides with the group of algebra homomorphism $H \longrightarrow k$. The operation of $G(H^*)$ is the convolution

$$(\lambda * \mu)(h) := \sum_{(h)} \lambda(h_{(1)})\mu(h_{(2)}) \qquad \forall\, \lambda,\, \mu \in G(H^*).$$

Note that $G(H^*)$ is commutative, whenever H is cocommutative. If $H = H(\mathcal{G})$ is the Hopf algebra of a finite algebraic k-group, then $G(H(\mathcal{G})^*) = G(\mathcal{O}(\mathcal{G}))$ is the set of group-like elements of the function algebra $\mathcal{O}(\mathcal{G})$ of $\mathcal{G}$.

Given $\lambda \in G(H^*)$ we observe that the map

$$\lambda * id_H : H \longrightarrow H \quad ; \quad h \mapsto \sum_{(h)} \lambda(h_{(1)}) h_{(2)}$$

is an automorphism, whose inverse is $(\lambda \circ \eta) * id_H$. If M is an H-module, we will abuse notation and write $M^{(\lambda)}$ rather than $M^{(\lambda * id_H)}$. In the special case, where $M = k$ is the trivial H-module, we put $k_\lambda := k^{(\lambda\circ\eta)}$, so that

$$h \cdot 1 = \lambda(h)1 \qquad \forall\, h \in H.$$