SAMPLE PATH LARGE DEVIATIONS IN FINER TOPOLOGIES

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Abstract. In this paper we present sufficient conditions for sample path large deviation principles to be extended to finer topologies. We consider extensions of the uniform topology by Orlicz functionals and we consider Lipschitz spaces: the former are concerned with cumulative path behaviour while the latter are more sensitive to extremes in local variation. We also consider sample paths indexed by the half line, where the usual projective limit topologies are not strong enough for many applications.

1. Introduction

Let Z be a topological space. A good rate function is a lower semi-continuous function $I : Z \to \mathbb{R}_+$, with compact level sets $\{z : I(z) \leq \alpha\}$. A sequence of random variables $(Z_n)_{n\in\mathbb{N}}$, taking values in Z, is said to satisfy the large deviation principle (LDP) with rate function I if for all closed $B \subset Z$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n \in B) \leq -\inf_{z\in B} I(z),$$

and for all open $G \subset Z$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n \in G) \geq -\inf_{z\in B} I(z).$$

Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of random variables with values in $\mathbb{R}^d$. For each $0 \leq t \leq 1$ set

$$S_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_k.$$

Denote by $L_\infty[0,1]$ the space of (equivalence classes modulo equality *a.e.* of) bounded measurable functions on $[0,1]$, equipped with the uniform topology, by $\mathcal{C}[0,1]$ the subspace of continuous functions, and by $\mathcal{A}[0,1]$ the subspace of absolutely continuous functions ϕ on $[0,1]$ with $\phi(0) = 0$. Denote by $L^d_\infty[0,1]$ the space of bounded measurable functions on $[0,1]$ with values in $\mathbb{R}^d$, equipped with the uniform topology.

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It is a classical result, originally due to Varadhan [13] and Mogulskii [9], that, if the $(X_k)_{k\in\mathbb{N}}$ are *i.i.d.* and

$$\Lambda(\theta) := \log \mathbb{E}\, \exp(\langle \theta, X_1 \rangle) < \infty \tag{1.1}$$

for all $\theta \in \mathbb{R}^d$, then the sequence $\{S_n(\cdot), n \in \mathbb{N}\}$ satisfies the LDP in $L^d_\infty[0,1]$ with good rate function given by

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi})ds & \phi \in \mathcal{A}^d[0,1], \\ \infty & \text{otherwise}, \end{cases} \tag{1.2}$$

where Λ^* is the convex dual of Λ:

$$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}^d} [\langle x, \theta \rangle - \Lambda(\theta)],$$

and where $\dot{\phi}$ denotes the derivative of ϕ, which exists almost everywhere for $\phi \in \mathcal{A}^d[0,1]$. The assumption that the moment generating function is finite has since been relaxed by Pinelis [11]; more recently, Dembo and Zajic [3] have relaxed the assumption of independence, replacing it with a variety of mixing and tightness hypotheses.

We find it convenient to consider, instead of S_n, its polygonal approximation:

$$\tilde{S}_n(t) = S_n(t) + \left(t - \frac{[nt]}{n}\right) X_{[nt]+1}. \tag{1.3}$$

Note that $\tilde{S}_n(\cdot)$ is continuous, and carries the same information as $S_n(\cdot)$. The LDP for $\tilde{S}_n(\cdot)$ can typically be shown to hold under milder hypotheses, and in the Polish space $\mathcal{C}^d[0,1]$ (equipped with the uniform topology).

In this paper we present sufficient conditions for such an LDP to be extended to finer topologies. In Section 3 we consider extensions of the uniform topology by Orlicz functionals and in Section 4 we consider Lipschitz spaces; the former are concerned with cumulative path behaviour while the latter are more sensitive to extremes in local variation. We also consider sample paths indexed by the half line, where the usual projective limit topologies are not strong enough for many applications, particularly in queueing theory. This extends work of O'Connell [10] and work of Dobrushin and Pechersky [5].

2. Some facts about Orlicz spaces

What follows is mostly a summary of the relevant details from [1, Chapter 8]. A function $C : \mathbb{R}_+ \to \mathbb{R}_+$ is called an *N-function* if it is convex and satisfies

$$\lim_{x\to 0^+} C(x)/x = 0, \qquad \lim_{x\to\infty} C(x)/x = \infty.$$

Examples of N-functions are

$$C(x) = x^p, \quad (1 < p < \infty),$$

$$C(x) = e^x - x - 1,$$