MULTIDIMENSIONAL VERSION OF A RESULT OF SAKHANENKO IN THE INVARIANCE PRINCIPLE FOR VECTORS WITH FINITE EXPONENTIAL MOMENTS

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Abstract. A multidimensional version of a result of Sakhanenko for the Gaussian approximation of sequences of successive sums of independent non-identically distributed random vectors with finite exponential moments is obtained.

1. Introduction

The aim of this paper is to give a multidimensional generalization of a result of Sakhanenko (1984) which is an extension and sharpening of the famous result of Komlós, Major and Tusnády (KMT) (1975–76) to the case of non-identically distributed random variables. For formulations of results we need the following notation.

Notation 1.1. We write $z \in \mathbb{R}^d$ (or $\mathbb{C}^d$), if $z = (z_1, \dots, z_d) = z_1 e_1 + \dots + z_d e_d$, where $z_j \in \mathbb{R}^1$ (or $\mathbb{C}^1$) and the e_j are the basis vectors. The scalar product of vectors $x, y \in \mathbb{R}^d$ (or $\mathbb{C}^d$) is denoted by $\langle x, y \rangle = x_1 \overline{y}_1 + \dots + x_d \overline{y}_d$. For $z \in \mathbb{R}^d$ (or $\mathbb{C}^d$) we use the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$ and the maximum norm $|z| = \max\limits_{1 \leqslant j \leqslant k} |z_j|$. For $b > 0$ we write $\log^* b = \max \{1, \log b\}$. The distribution and the corresponding

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covariance operator of a random vector ξ is denoted by $\mathcal{L}(\xi)$ and $\operatorname{cov}\xi$ (or $\operatorname{cov} F$, if $F = \mathcal{L}(\xi)$). For random vectors $X_1, \dots, X_s$ and $Y_1, \dots, Y_s$ we write

$$\Delta(X, Y) = \max_{1 \leqslant j \leqslant s} \Big\| \sum_{i=1}^{j} X_i - \sum_{i=1}^{j} Y_i \Big\|. \tag{1.1}$$

The symbol $\mathbb{I}_d : \mathbb{R}^d \to \mathbb{R}^d$ is used for the identity operator in $\mathbb{R}^d$. Introduce the projectors $\mathbb{P}_i : \mathbb{R}^d \to \mathbb{R}^1$ and $\overline{\mathbb{P}}_j : \mathbb{R}^d \to \mathbb{R}^j$, $i, j = 1, \dots, d$, by the relations $\mathbb{P}_i x = x_i$, $\overline{\mathbb{P}}_j x = (x_1, \dots, x_j)$, where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.

The rate for strong approximation in the one-dimensional invariance principle was studied by many authors (see, e.g., Prokhorov (1956), Skorokhod (1961), Borovkov (1973), Csörgő and Révész (1975) and the bibliography in Csörgő and Révész (1981), Csörgő and Hall (1984), Shao (1995)). Skorokhod (1961) developed a method of construction of close sequences of sequential sums of independent random variables on the same probability space. For a long time the best rates of approximation were obtained by this method, known now as the Skorokhod embedding. However, Komlós, Major and Tusnády (KMT) (1975–76) elaborated a new, more powerful method of dyadic approximation. With the help of this method they obtained optimal rates of Gaussian approximation for sequences of independent identically distributed random variables.

Sakhanenko (1984) generalized and essentially sharpened KMT results in the case of non-identically distributed random variables. He considered the following class of one-dimensional distributions:

$$\mathcal{S}_1(\tau) = \Big\{ \mathcal{L}(\xi) : \mathbf{E}\,\xi = 0,\ \ \mathbf{E}\,|\xi|^3 \exp\big(\tau^{-1}|\xi|\big) \leqslant \tau\,\mathbf{E}\,|\xi|^2 \Big\}. \tag{1.2}$$

His main result is formulated as follows.

Theorem 1.2 (Sakhanenko, 1984). *Suppose that $\tau > 0$, and $\xi_1, \dots, \xi_s$ are independent random variables with $\mathcal{L}(\xi_j) \in \mathcal{S}_1(\tau)$, $j = 1, \dots, s$. Then one can construct on a probability space a sequence of independent random vectors $X_1, \dots, X_s$ and a sequence of independent Gaussian random vectors $Y_1, \dots, Y_s$ so that $\mathcal{L}(X_j) = \mathcal{L}(\xi_j)$, $\mathbf{E}\,Y_j = 0$, $\mathbf{E}\,Y_j^2 = \mathbf{E}\,X_j^2$, $j = 1, \dots, s$, and*

$$\mathbf{E}\,\exp\big(c_1\,\Delta(X, Y)/\tau\big) \leqslant 1 + B/\tau, \tag{1.3}$$

where c_1 is an absolute constant and $B^2 = \mathbf{E}\,\xi_1^2 + \dots + \mathbf{E}\,\xi_s^2$.

Multidimensional estimates in the strong invariance principle can be found in Berkes and Philipp (1979), Philipp (1979), Berger (1982), Einmahl (1987a,b; 1989). Zaitsev (1995, 1996, 1998) removed an unnecessary logarithmic factor from the result of Einmahl (1989) and obtained multidimensional analogs of KMT results. In Zaitsev (1998, Theorem 1.3) the random vectors are, generally speaking, non-identically