AUSLANDER-REITEN COMPONENTS FOR LIE ALGEBRAS OF REDUCTIVE GROUPS

Rolf Farnsteiner*

0. Introduction

In this paper we shall study nonperiodic connected components of the stable Auslander-Reiten quiver of a reduced enveloping algebra $u(L,\chi)$ associated to a restricted Lie algebra $(L,[p])$. According to earlier results (cf. [13, 15]) the tree classes of these AR-components are either Euclidean diagrams, or the infinite Dynkin diagrams A_∞, D_∞, A_∞^∞. In the context of finite groups K. Erdmann has recently shown that the latter two trees cannot occur for wild blocks (cf. [12]). As demonstrated in [13, 18], a similar result holds for the restricted enveloping algebra $u(L) := u(L,0)$ in case the underlying Lie algebra L is nilpotent and of characteristic $p \geq 3$. Here neither tame blocks nor components of tree class A_∞^∞ or D_∞ exist. For $p = 2$ the restricted enveloping algebra of the 2-unipotent Heisenberg algebra possesses components of tree class A_∞^∞. Presently, no Lie algebra admitting components of tree class D_∞ is known, and recent work [19] suggests that such components will be rather exceptional.

In default of a general block theory for enveloping algebras one is led to either employ geometric techniques (cf. [19]), or to focus on those cases, where the block structure of $u(L,\chi)$ is governed by well-understood "linkage principles" (cf. [13, 17, 18]). The former are most effective for Lie algebras of algebraic groups, while the latter for instance play a rôle in the AR-theory of supersolvable Lie algebras.

Our paper can roughly be divided into two parts. The purpose of sections 1 through 3 is to furnish basic properties of components of infinite Dynkin type. By using a modification of a recognition criterion due to K. Erdmann [12], we provide in section 2 general results pertaining to the aforementioned components. Components of tree class A_∞ are discussed in section 3. In particular, we show that such a component contains at most one module of length 2.

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The second part combines these results with geometric techniques in case $L = Lie(G)$ is the Lie algebra of an algebraic group G. In this setting the adjoint representation $Ad : G \longrightarrow GL(L)$ induces for every linear form $\chi \in L^*$ an action of its stabilizer G_χ on the set of stable Auslander-Reiten components of $u(L,\chi)$. Following some technical preparations, we turn in section 5 to the study of AR-components of restricted enveloping algebras for Lie algebras of reductive groups. We focus on invariant components, i.e., those that are fixed by the above action. In view of [20, (2.2)] these are precisely those components whose vertices are Ad-stable in the sense that twisting of the module by the adjoint representation yields an isomorphic module. Components of Euclidean tree class, and those containing indecomposable constituents of rational G-modules belong to this class. Our main results, Theorems 5.2 and 5.5, show that invariant components are of type $\mathbb{Z}[A_\infty]$ or $\mathbb{Z}[\tilde{A}_{12}]$, with the latter occurring only if $SL(2)$ or $PSL(2)$ is a direct factor of G. In particular, $\mathbb{Z}[\tilde{A}_{12}]$ is the only component of Euclidean tree class, a result that contrasts with the absence of such components for group algebras of finite groups of characteristic $p \geq 3$ (cf. [40]). The same methods afford the determination of those blocks of the restricted enveloping algebras of the Lie algebras of reductive groups that possess a simple module of complexity 2 (cf. (5.2)). These turn out to be full matrix algebras over blocks of $u(s\ell(2))$, thereby providing in this context a refinement of Voigt's work (cf. [51]) on tame enveloping algebras. In view of the Morita equivalence given in [26], our results continue to hold for reduced enveloping algebras whose defining linear forms belong to the Zariski dense subset of semisimple elements. However, for the so-called nilpotent linear forms different phenomena are known to occur (cf. [17, 44]).

In the final section we illustrate how recent results by Suslin, Friedlander and Bendel [49] may be employed to study blocks of higher Frobenius kernels of reductive groups. Although some important features of the theory no longer hold in this context, one can still determine the Morita equivalence classes of the tame blocks (cf. (7.1)).

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1. Preliminaries

The purpose of this preparatory section is to modify various ideas of [12] to obtain information concerning modules belonging to components of tree classes A_∞^∞, D_∞ and $\tilde{D}_n$.

Let Λ be a Frobenius algebra, defined over a field F, with Nakayama automorphism $\mu : \Lambda \longrightarrow \Lambda$. Given a Λ-module M and an automorphism $\gamma \in Aut(\Lambda)$, we let $M^{(\gamma)}$ be the module with underlying vector space M and action

$$a \cdot m := \gamma^{-1}(a)\, m \quad \forall\, a \in \Lambda,\ m \in M.$$

For $i \in \mathbb{Z}$, we put $M^{(i)} := M^{(\mu^i)}$. Recall that the stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$ has the isomorphism types $[M]$ of nonprojective indecomposable Λ-modules as its vertices. By