DERIVED DYNKIN EXTENSIONS

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1. Introduction and main result

1.1. Let k be an algebraically closed field. For basic notation and results we refer to [GR] and [Ri]. All algebras which we will consider are finite-dimensional basic algebras over k. Sometimes we will consider such an algebra as a finite spectroid. For an algebra A we denote by mod$-A$ the category of finite-dimensional right A-modules and by ind$-A$ a spectroid of this aggregate. If A is considered as spectroid, then mod$-A$ is identified with the category of contravariant functors from A to the category of finite-dimensional k-spaces.

It is well-known that for any extended Dynkin diagram $\widetilde{\Delta}$ the path algebra kQ is hereditary and of tame representation type where Q is a quiver whose underlying graph is $\widetilde{\Delta}$. Recall that the extended Dynkin diagrams are $\widetilde{\mathbb{A}}_n$ $(n \geq 1)$, $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$. Obviously, to each extended Dynkin diagram $\widetilde{\Delta}$ there corresponds a unique Dynkin diagram Δ in the list $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ obtained by cancelling an extremal vertex.

The Auslander-Reiten quiver of the algebra $H = kQ$ consists of a preprojective component $\mathcal{P}$, a preinjective component $\mathcal{I}$ and a family $\mathcal{T} = (\mathcal{T}_\varrho)_{\varrho \in \mathbb{P}(k)}$ of stable tubes. For each of these components we denote by the same symbol the induced full subcategory of ind$-H$ resp. mod$-H$. In particular, each of the tubes $\mathcal{T}_\varrho$ is an abelian subcategory of mod$-H$ with simple modules $E_1, \dots, E_{r(\varrho)}$. The positive integer $r(\varrho)$ is called the rank of the tube $\mathcal{T}_\varrho$. The map $r : \mathbb{P}(k) \to \mathbb{N}$ is said to be the tubular rank function of H. From the tubular rank function we form a graph by considering the collection $(\mathbb{A}_{r(\varrho)})_{\varrho \in \mathbb{P}(k)}$, choosing an extremal vertex in all these linear graphs and glueing them together in the chosen vertices. This star is said to be the tubular type of H. Since most of the tubes $\mathcal{T}_\varrho$ are homogeneous i.e. $r(\varrho) = 1$, the tubular type is actually a finite graph. It turns out that the tubular type of a tame hereditary algebra $H = kQ$ such that Q is a quiver with underlying extended Dynkin diagram $\widetilde{\Delta}$ is the associated Dynkin diagram Δ.

1.2. For a finite-dimensional algebra A we denote by $D^b(A)$ the derived category of bounded complexes in mod$-A$ and by ind$(D^b(A))$ a spectroid of this aggregate. The algebras A such that there is a triangle equivalence $F : D^b(A) \to D^b(H)$ where H is a tame hereditary algebra of the form $H = kQ$ are called piecewise tame hereditary of type Q.

Since H is hereditary the category ind$(D^b(H))$ is the disjoint union of all the shifts ind$-H[\nu]$ $(\nu \in \mathbb{Z})$ where mod$-H$ is included into $D^b(H)$ in the canonical way as complexes concentrated in degree 0. More precisely, the Auslander-Reiten quiver of $D^b(H)$ consists of all the shifts $\mathcal{T}[\nu]$ of the tubular family $(\mathcal{T}_\varrho)_{\varrho \in \mathbb{P}(k)}$ and directed components $\mathcal{C}[\nu]$ obtained by glueing together $\mathcal{I}[\nu - 1]$ and $\mathcal{P}[\nu]$ at the injective resp. projective vertices.

The most simple examples of the piecewise tame hereditary algebras are the tame concealed algebras where the triangle equivalence F is induced by a preprojective tilting module. In particular, the tame concealed algebras were classified in [HV].

1.3. Recall that for a given finite-dimensional algebra B and a module R in $B-$mod the one-point extension algebra $B[R]$ is by definition the matrix algebra $\begin{pmatrix} k & R \\ 0 & B \end{pmatrix}$ which is endowed with the usual addition and matrix multiplication. When considering $B[R]$ as a finite spectroid the one-point extension proecess means the creating of a new object which we usually call ω. In this interpretation the morphism space from an object x of B to ω is $R(x)$, there are no morphisms from ω to x and the endomorphism space of ω is k.

We will also need the dual construction. The one-point coextension algebra $[R]B$ is defined as $\begin{pmatrix} B & \mathrm{D}(R) \\ 0 & k \end{pmatrix}$ where $D := \mathrm{Hom}_k(-,k)$ is the usual duality mod$-B \to B-$mod from the finite-dimensional right to the finite-dimensional left modules.

If now B is a piecewise tame hereditary algebra of type Q and $F : D^b(B) \to D^b(kQ)$ is a triangle equivalence, then we consider a B-module R such that $F(R)$ is a simple object in a shifted tube $\mathcal{T}_\sigma[\nu]$ for some $\sigma \in \mathbb{P}(k)$ and some $\nu \in \mathbb{Z}$. Given the tubular rank function r of kQ we define the *derived tubular extension rank function* of the one-point extension $B[R]$ and also the one-point coextension $[R]B$ as the map r' such that $r'(\varrho) = r(\varrho)$ for all $\varrho \neq \sigma$ and $r'(\sigma) = r(\sigma) + 1$. The star associated with r' as above will be called the derived tubular extension type of $B[R]$ and also of $[R]B$. We say that A is a *derived Dynkin extension* of B by R if A can be identified with $B[R]$ or $[R]B$ and the derived tubular extension type of A is again a Dynkin diagram.

1.4. In the next section we will recall a result of Barot and Lenzing (see [BL]) which allows to extend triangle equivalences between derived categories of algebras to triangle equivalences of derived categories of certain one-point extensions. This will show that a derived Dynkin extension of a piecewise tame hereditary algebra is again piecewise tame hereditary. This allows to iterate the process of forming derived Dynkin extensions. More precisely, given a finite-dimensional algebra A and a piecewise tame hereditary algebra B, we say that A is an *iterated derived Dynkin extension* of B if there is a non-negative integer n, a sequence $B = B_0, \ldots, B_n = A$ of algebras and a sequence $R_0, \ldots, R_{n-1}$ of modules R_i in mod$-B_i$ such that for all $i = 1, \ldots, n$ the algebra B_i is a derived Dynkin extension of B_{i-1} by R_{i-1}. In particular, the derived tubular extension type of A is well-defined and a Dynkin diagram.

With this notation we can formulate the main theorem of this paper:

Theorem. *Let A be a finite-dimensional k-algebra and Δ be a Dynkin diagram. Then A is a representation infinite piecewise tame hereditary algebra of tubular type Δ if and only if there is a tame concealed algebra B such that A is an iterated derived Dynkin extension of B of derived tubular extension type Δ.*

We will prove this result at the end of section 2. In section 3 we will use it to reprove a characterisation due to Assem and Skowroński (see [AS]) saying that the representation infinite piecewise tame hereditary algebras are precisely the domestic tubular branch enlargements of the tame concealed algebras.

The investigations of this paper were motivated by [DS] were tame multiple one-point extensions by modules R over piecewise tame hereditary algebras which are sent by F to