The numerical computation and geometrical analysis of heteroclinic tangencies

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Abstract

Quadratic tangential homoclinics, also called homoclinic bifurcations, are known to cause highly complex behavior of time–discrete dynamical systems. In this paper we develop a method to approximate these orbits. More generally, we consider the case of tangential heteroclinics. These orbits are approximated by finite orbit–segments solving a boundary value problem and satisfying a regularity constraint. A detailed error analysis for this method is given which includes the case of local branches of heteroclinic orbits. We apply our method to the computation of quadratic tangentials. They are approximated as turning points of the boundary value operator. Several examples illustrate this approach. Furthermore, equivalences are derived between geometric properties of the intersection of the stable and unstable manifolds at a heteroclinic point on one hand, and analytic properties of the variational equation along the heteroclinic orbit on the other hand.

Key words. dynamical systems – homoclinic bifurcations – heteroclinic orbits for maps – numerical methods

1 Introduction

One famous result on chaotic dynamics of time–discrete systems is the homoclinic theorem of Smale and Birkhoff, see [23], [22]. It states that in some neighborhood of a transversal homoclinic orbit there exists a Cantor set on which the dynamics of the system is conjugate to a subshift of finite type.

More complex phenomena occur near a homoclinic bifurcation which is a generic unfolding of a quadratic homoclinic tangency. For a class of one–parametric surface diffeomorphisms, Mora and Viana have shown in [15] that

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the existence of a homoclinic bifurcation at some parameter value, say $\bar{\lambda}$, implies the appearance of a strange attractor for every parameter value in some positive Lebesgue measure set near $\bar{\lambda}$. Other results on more general diffeomorphisms were published by Viana later on. A historical review of the discovery of homoclinic phenomena is given by Palis and Takens in [17]. This monograph collects results mainly from western literature. Earlier references to these topics in the east are the papers [8], [9] by Gavrilov and Shilnikov.

Consider a time–discrete system

$$x_{n+1} = f(x_n, \lambda), \quad n \in \mathbb{Z}, \tag{1}$$

where $f : \mathbb{R}^k \times \mathbb{R}^r \to \mathbb{R}^k$ is a $\mathcal{C}^l$–map with $l \geq 1$, and $f(\cdot, \lambda)$ is diffeomorphic for every λ in the open ball $U_\epsilon(\bar{\lambda})$ of some radius $\epsilon > 0$ centered at some $\bar{\lambda} \in \mathbb{R}^r$. In case $r = 0$ all assumptions on the parameter are neglected throughout this paper. Furthermore, let $\bar{\xi}_-$, $\bar{\xi}_+$ be *hyperbolic fixed points of* $f(\cdot, \bar{\lambda})$, i.e. $\bar{\xi}_\pm = f(\bar{\xi}_\pm, \bar{\lambda})$ and $f_x(\bar{\xi}_\pm, \bar{\lambda})$ has no eigenvalue on the unit circle. A sequence $(\bar{x}_n)_{n\in\mathbb{Z}} = \bar{x}_\mathbb{Z}$ that solves (1) for $\lambda = \bar{\lambda}$, i.e. a $f(\cdot, \bar{\lambda})$–*orbit*, is called *heteroclinic connecting* $\bar{\xi}_-$ *and* $\bar{\xi}_+$ if

$$\lim_{n\to-\infty} \bar{x}_n = \bar{\xi}_-, \quad \lim_{n\to+\infty} \bar{x}_n = \bar{\xi}_+ \tag{2}$$

hold. In case $\bar{\xi}_- = \bar{\xi}_+$ such an orbit is called *homoclinic with respect to* $\bar{\xi}_\pm$.

Obviously, every point $\bar{x}_n$, $n \in \mathbb{Z}$, of a heteroclinic $f(\cdot, \bar{\lambda})$–orbit $\bar{x}_\mathbb{Z}$ connecting $\bar{\xi}_-$ and $\bar{\xi}_+$ lies in the intersection of the *unstable manifold* $W_u(\bar{\xi}_-, \bar{\lambda}) = \{x : \lim_{n\to-\infty} f(\cdot, \bar{\lambda})^n(x) = \bar{\xi}_-\}$ and the *stable manifold* $W_s(\bar{\xi}_+, \bar{\lambda}) = \{x : \lim_{n\to+\infty} f(\cdot, \bar{\lambda})^n(x) = \bar{\xi}_+\}$. We call the heteroclinic orbit $\bar{x}_\mathbb{Z}$ t*–tangential* if the tangent spaces of these manifolds at $\bar{x}_0$ have a common subspace of dimension t. In case $t = 0$ (and $r = p + t$ where p is defined in (3)) it is called *transversal.* A transversal heteroclinic together with parts of the manifolds $W_u(\bar{\xi}_-, \bar{\lambda})$ and $W_s(\bar{\xi}_+, \bar{\lambda})$ is shown on the left side of Figure 1.

A 1–tangential heteroclinic is called *quadratic* if the manifolds $W_u(\bar{\xi}_-, \bar{\lambda})$ and $W_s(\bar{\xi}_+, \bar{\lambda})$ have a quadratic tangency at $\bar{x}_0$. Such an orbit, together with parts of these manifolds, is drawn on the right side of Figure 1.

Furthermore, consider the *extended unstable* and *stable manifolds*

$$\begin{aligned} \mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda}) &= \{(x, \lambda) : \lambda \in U_\varepsilon(\bar{\lambda}), x \in W_u(\xi_-(\lambda), \lambda)\}, \\ \mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda}) &= \{(x, \lambda) : \lambda \in U_\varepsilon(\bar{\lambda}), x \in W_s(\xi_+(\lambda), \lambda)\}. \end{aligned}$$

These sets are injectively $\mathcal{C}^l$–immersed submanifolds of $\mathbb{R}^{k+r}$, cf. Appendix B. Let

$$p = k_{-s} - k_{+s} \tag{3}$$

where $k_{\pm s}$ is the dimension of $W_s(\bar{\xi}_\pm, \bar{\lambda})$ and $\xi_\pm(\lambda)$ is the locally unique fixed point of $f(\cdot, \lambda)$ near $\bar{\xi}_\pm$ for $\lambda \in U_\epsilon(\bar{\lambda})$ and $\epsilon > 0$ sufficiently small. We call a t–tangential heteroclinic *nondegenerate* if $r = p + t$ holds and the extended manifolds $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda})$ are transversal at $(\bar{x}_0, \bar{\lambda})$. In Figure 2 we illustrate the intersection of $\mathcal{W}_{u,\epsilon}(\bar{\xi}_-, \bar{\lambda})$ and $\mathcal{W}_{s,\epsilon}(\bar{\xi}_+, \bar{\lambda})$ near $(\bar{x}_0, \bar{\lambda})$ in case of a nondegenerate quadratic heteroclinic $\bar{x}_\mathbb{Z}$.

In case $r = 1$, a generic unfolding of a quadratic heteroclinic tangency occurs exactly at a nondegenerate quadratic 1–tangential homoclinic. To compute such tangencies we will state and analyze a method for the approximation of