Simplicial functors and stable homotopy theory

Manos Lydakis
Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany
manos@math206.mathematik.uni-bielefeld.de

May 6, 1998

Summary— We study what we call the simplicial category of simplicial functors. We show that it has a model structure which is Quillen-equivalent to the model structure on the simplicial category of spectra, and that it has a symmetric monoidal product corresponding to the smash product of spectra. As a byproduct, we obtain simpler and more descriptive constructions of the model structures on spectra. As another byproduct, we obtain a model structure on simplicial functors which should be thought of as a model structure on homotopy functors. As a final byproduct, we obtain a model-category version of the part of Goodwillie's calculus of homotopy functors having to do with "the linear approximation of a homotopy functor at $*$" [G1], without the restriction on the relevant homotopy functors of being "stably excisive".

1. Introduction

The problem of constructing a nice smash product of spectra is an old and well-known problem of algebraic topology. This problem has come to mean the following: Find a model category, which is Quillen-equivalent to the model category of spectra, and which has a symmetric monoidal product corresponding to the smash product of spectra. Two solutions were found recently, namely the smash product of S-modules of [EKMM], and the smash product of symmetric spectra of [HSS]. Here we present another solution, the smash product of simplicial functors. As a byproduct, we obtain simpler and more descriptive constructions of the model structures on spectra. As another byproduct, we obtain a model structure on simplicial functors which should be thought of as a model structure on homotopy functors. As a final byproduct, we obtain a model-category version of the part of Goodwillie's calculus of homotopy functors having to do with "the linear approximation of a homotopy functor at $*$" [G1], without the restriction on the relevant homotopy functors of being "stably excisive".

We say a few words about particular features that distinguish simplicial functors from S-modules and symmetric spectra. First of all, an advantage of simplicial functors is their simplicity. Also, the monoids under the smash product of simplicial functors (these are what one would call "algebras over the sphere spectrum", after thinking of simplicial functors as spectra by using the results of this paper) are well known: They are the FSPs, which were introduced in 1985 by Bökstedt [B]. There are interesting constructions with FSPs, e.g., topological cyclic homology, which can be considerably simplified and conceptualized using the smash product of simplicial functors [LS]. Note however that it is not clear that an E_∞-FSP has a commutative model, although the analogous statement is true for S-modules and symmetric spectra. This has, e.g., the disadvantage, that it is not clear how to give a model category structure to (or even how to define) MU-algebras using simplicial functors (although this can be done for modules over any any FSP, and algebras over any commutative FSP, by similar methods as in [Sch]). Returning to examining the special features of simplicial functors, in constrast to S-modules and symmetric spectra, simplicial functors have not only an interesting unstable theory (modeling certain homotopy functors, cf. remark 4.6 and section 8), but interesting "metastable", as well as a whole tower of "higher stable" theories, which are related to the calculus of homotopy functors [G1], [G2], [G3] (this will appear in a future paper). Finally, simplicial functors can be not only smashed, but also composed, with each other. There is a canonical map from the smash product to the composition product, which is an isomorphism in an important special case (proposition 5.13), a fact that provides a very concrete description of the smash product. There is a close connection between simplicial functors and Γ-spaces [L2]. In fact, Γ-spaces may be identified with a special kind of simplicial functors (see convention 2.11 of [L2]) in such a way that the smash product of Γ-spaces corresponds to the smash product of simplicial functors under this identification.

We have tried to make this paper reasonably self-contained. We develop from scratch the part of the theories of simplicial categories, of model categories, and of stable homotopy, that we need. In particular, no background is assumed from the reader, other than familiarity with simplicial sets (and some basic algebraic topology).