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Spanning subgraphs of random graphs

OLIVER RIORDAN

SFB 343, Universität Bielefeld
Postfach 10 01 31, 33501 Bielefeld, Germany

Department of Pure Mathematics and Mathematical Statistics
16 Mill Lane, Cambridge CB2 1SB, England

Let G_p be a random graph on 2^d vertices where edges are selected independently with a fixed probability $p > \frac{1}{4}$, and let H be the d-dimensional hypercube Q^d. We answer a question of Bollobás by showing that, as $d \to \infty$, G_p almost surely has a spanning subgraph isomorphic to H. In fact we prove a result which can be applied to many other graphs, improving also previous results for the lattice, i.e., the 2-dimensional square grid. The proof uses the second moment method—writing X for the number of subgraphs of G isomorphic to H, where G is a suitable random graph, we expand the variance of X as a sum over all subgraphs of H itself. As the subgraphs of H may be quite complicated, most of the work is in estimating the various terms of this sum.

1. Introduction

In this paper we consider the following general question: when is a random graph G likely to have a spanning subgraph isomorphic to a given graph H? For the graph H we concentrate on two special cases, that of the *cube* and that of the *lattice*, although we prove a more general result which covers both these cases as well as many others. For the random graph G we consider the two standard models $\mathcal{G}(n,p)$ and $\mathcal{G}(n,M)$. As usual, a random $G \in \mathcal{G}(n,p)$ is a graph with vertex set $[n] = \{1, 2, \dots, n\}$ obtained by selecting each possible edge independently with probability p. Writing $N = \binom{n}{2}$, a random $G \in \mathcal{G}(n,M)$ is obtained by selecting each of the $\binom{N}{M}$ graphs on $[n]$ with M edges with equal probability.

Let Q^d, the *d-cube*, be the graph with vertex set $\{0,1\}^d$ where two sequences are adjacent if they differ in precisely one place. The following question due to Bollobás (see [1]) provides the motivation for this paper: for which p does $G \in \mathcal{G}(2^d, p)$ almost surely contain a copy of Q^d?

Alon and Füredi [1] gave a partial answer to this question, showing that any fixed $p > \frac{1}{2}$ suffices. They also noted that for $p = \frac{1}{4}$ the expected number of d-cubes in G

tends to zero, so $p > \frac{1}{4}$ is definitely necessary. We shall prove the following result, which is thus close to best possible.

Theorem 1. *Let $p = p(d) = \frac{1}{4} + 6\log d/d$. Then with probability tending to 1 as $d \to \infty$ a random $G \in \mathcal{G}(2^d, p)$ contains a copy of Q^d.*

Another graph for which such a question has been asked is the *lattice* L_k, the graph on $[k] \times [k]$ in which two points are adjacent if the Euclidean distance between them is 1. The question of which random graphs contain spanning lattices was raised by Venkatesan and Levin [8]. Writing $n = k^2$, Alon and Füredi [1] showed that $p(n) = O((\log n/n)^{1/4})$ will do, noting that $p = n^{-1/2}$ is a lower bound. We shall show that the lower bound is again essentially correct, proving the following result.

Theorem 2. *Let $p = p(n) = \omega(n)n^{-1/2}$, where $\omega(n)$ tends to infinity arbitrarily slowly, and let $n = k^2$. Then with probability tending to 1 as $k \to \infty$ a random $G \in \mathcal{G}(n,p)$ contains a copy of L_k.*

Note that, together with the lower bound given by the expectation, Theorem 2 states that $p = n^{-1/2}$ is a threshold for the property of containing a spanning lattice, in the sense of Erdős and Rényi (see [2]).

Theorem 2 also strengthens a result of Fernandez de la Vega and Manoussakis [5] concerning lattices L_k in $G \in \mathcal{G}(n,p)$ where k^2 is some constant fraction of n less than one, which we return to later.

We shall deduce Theorems 1 and 2 from a common generalisation which could also be applied to many other graphs. As the statement of this result is rather cumbersome, we postpone it to the next section. We note here, however, that we shall be forced to work in $\mathcal{G}(n, M)$ rather than $\mathcal{G}(n,p)$. We shall thus derive equivalents of Theorems 1 and 2 for this model, namely the following. As before we write N for $\binom{n}{2}$.

Theorem 3. *Let $p' = p'(d) = \frac{1}{4} + 5\log d/d$, and let $n = 2^d$. Then with probability tending to 1 as $d \to \infty$ a random $G \in \mathcal{G}(n, p'N)$ contains a copy of Q^d.*

Theorem 4. *Let $p' = p'(n) = \omega'(n)n^{-1/2}$, where $\omega'(n) \to \infty$ arbitrarily slowly, and let $n = k^2$. Then with probability tending to 1 as $k \to \infty$ a random $G \in \mathcal{G}(n, p'N)$ contains a copy of L_k.*

Note that these results immediately imply Theorems 1 and 2, as a random $G \in \mathcal{G}(n,p)$ almost certainly has at least $p'N$ edges, whenever $N(p-p')/\sqrt{Np(1-p)} \to \infty$. In the case of Theorem 1, this condition is satisfied by the given p and p'; for Theorem 2 we just take $\omega'(n) = \omega(n)/2$, for example.

Before getting down to work, we give a very brief outline of the method used. Previous approaches ([1], [5]) to the above questions involved finding a copy of $H = Q^d$ or L_k in G one piece at a time. In contrast, we look at the random variable X, the number of full copies of H in G. We bound the variance of X in terms of its expectation, and use