# Superconvergence estimates for the numerical computation of heteroclinics for maps

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**Abstract**

In [5] a method for the approximation of heteroclinic orbits in parameterized time–discrete systems is given which applies both to transversal and quadratic tangential heteroclinics. Solutions of a boundary value problem are used to approximate the heteroclinic on some finite interval as well as the parameter. It turns out that the error in the parameter decays much faster than that one in the orbit. In this paper we support these numerical findings by proving superconvergence estimates for the error in the parameter. Examples show that these estimates are sharp in case of projection boundary conditions but not in case of periodic boundary conditions.

**Key words.** dynamical systems – homoclinic bifurcations – heteroclinic orbits for maps – numerical methods – convergence estimates

**AMS subject classifications.** 65L12, 58F08

## 1 Introduction

We consider a parameterized time–discrete system

$$x_{n+1} = f(x_n, \lambda), \quad \lambda \in \mathbb{R}^r, \quad n \in \mathbb{Z}, \tag{1}$$

where $f \in \mathcal{C}^{l+1}(\mathbb{R}^k \times \mathbb{R}^r, \mathbb{R}^k)$ and $l \geq 2$. In [5] we consider the approximation of tangential heteroclinic orbits by finite boundary value problems. The purpose of this paper is to prove superconvergence of the parameter. We first review the basic setting as well as the results from [5].

Let $\bar{x}_\mathbb{Z} = (\bar{x}_n)_{n\in\mathbb{Z}}$ be a *heteroclinic $f(\cdot,\bar{\lambda})$–orbit connecting the hyperbolic fixed points $\bar{\xi}_-$ and $\bar{\xi}_+$ of $f(\cdot,\bar{\lambda})$*. This means that $(\bar{x}_\mathbb{Z}, \bar{\lambda})$ is a solution of (1) satisfying

$$\lim_{n\to\pm\infty} \bar{x}_n = \bar{\xi}_\pm$$

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where $\bar\xi_\pm = f(\bar\xi_\pm,\bar\lambda)$ and $f_x(\bar\xi_\pm,\bar\lambda)$ has no eigenvalue on the unit circle. In case $\bar\xi_- = \bar\xi_+$ the orbit is called *homoclinic.*

We assume that $f(\cdot,\lambda)$ is a diffeomorphism for every $\lambda \in U_\epsilon(\bar\lambda)$ and some $\epsilon > 0$. Let

$$\begin{aligned} W_u(\bar\xi_-,\bar\lambda) &= \{x \in \mathbb{R}^k : \lim_{n\to-\infty} f(\cdot,\bar\lambda)^n(x) = \bar\xi_-\}, \\ W_s(\bar\xi_+,\bar\lambda) &= \{x \in \mathbb{R}^k : \lim_{n\to+\infty} f(\cdot,\bar\lambda)^n(x) = \bar\xi_+\} \end{aligned}$$

be the unstable and stable manifold of $\bar\xi_-$ and $\bar\xi_+$ with respect to $f(\cdot,\bar\lambda)$. It is well known that these sets are injectively $\mathcal{C}^l$–immersed submanifolds. Furthermore, $\dim W_u(\bar\xi_-,\bar\lambda) = k_{-u} = k - k_{-s}$ and $\dim W_s(\bar\xi_+,\bar\lambda) = k_{+s} = k - k_{+u}$ hold where $k_{\pm s}$ and $k_{\pm u}$ denote the dimension of the stable and unstable subspace of $f_x(\bar\xi_\pm,\bar\lambda)$. We set

$$p = k_{-s} - k_{+s}. \tag{2}$$

By definition every point $\bar x_n$, $n \in \mathbb{Z}$, of the heteroclinic orbit $\bar x_\mathbb{Z}$ lies in the intersection of $W_u(\bar\xi_-,\bar\lambda)$ and $W_s(\bar\xi_+,\bar\lambda)$. We call $\bar x_\mathbb{Z}$ *$t$–tangential* if the tangent spaces $T_{\bar x_0}W_u(\bar\xi_-,\bar\lambda)$ and $T_{\bar x_0}W_s(\bar\xi_+,\bar\lambda)$ intersect along a subspace of dimension $t \geq 0$. This property can be characterized in terms of the $\mathcal{C}^l$–operator $\Gamma : S_Z \times \mathbb{R}^r \to S_\mathbb{Z}$ defined by

$$\Gamma(x_\mathbb{Z},\lambda) = (x_{n+1} - f(x_n,\lambda))_{n\in\mathbb{Z}}, \quad (x_\mathbb{Z},\lambda) \in S_\mathbb{Z} \times \mathbb{R}^r,$$

where $S_\mathbb{Z}$ is the Banach space of all bounded sequences in $\mathbb{R}^k$ indexed by $\mathbb{Z}$ and endowed with the supremum norm $\|x_\mathbb{Z}\|_\infty = \sup_{n\in\mathbb{Z}} \|x_n\|$. It turns out that the $f(\cdot,\bar\lambda)$–heteroclinic $\bar x_\mathbb{Z}$ is $t$–tangential if and only if the nullspace $\mathcal{N}(\Gamma_x(\bar x_\mathbb{Z},\bar\lambda))$ is of dimension $t$, i.e.

$$\dim \mathcal{N}(\Gamma_x(\bar x_\mathbb{Z},\bar\lambda)) = t. \tag{3}$$

Here $\Gamma_x$ denotes the partial derivative of $\Gamma$ with respect to the first variable.

To stabilize a tangential orbit we assume that $r = p + t$ holds and that the extended manifolds

$$\begin{aligned} \mathcal{W}_{u,\epsilon}(\bar\xi_-,\bar\lambda) &= \{(x,\lambda) : \lambda \in U_\epsilon(\bar\lambda), x \in W_u(\xi_-(\lambda),\lambda)\}, \\ \mathcal{W}_{s,\epsilon}(\bar\xi_+,\bar\lambda) &= \{(x,\lambda) : \lambda \in U_\epsilon(\bar\lambda), x \in W_s(\xi_+(\lambda),\lambda)\} \end{aligned}$$

are transversal at their common point $(\bar x_0,\bar\lambda)$, i.e.

$$T_{(\bar x_0,\bar\lambda)}\mathcal{W}_{u,\epsilon}(\bar\xi_-,\bar\lambda) + T_{(\bar x_0,\bar\lambda)}\mathcal{W}_{s,\epsilon}(\bar\xi_+,\bar\lambda) = \mathbb{R}^{k+r}.$$

By $\xi_\pm(\lambda)$, $\lambda \in U_\epsilon(\bar\lambda)$ we denote a fixed point of $f(\cdot,\lambda)$ close to $\bar\xi_\pm$. Taking $\epsilon > 0$ sufficiently small, this fixed point is locally unique, hyperbolic and $\xi_\pm$ is as smooth as $f$. A tangential heteroclinic satisfying both assumptions is called *nondegenerate.*

In [5] we approximate nondegenerate $t$–tangential heteroclinics as follows. For $-n_-, n_+ \in \mathbb{N}$ we consider solutions of the finite boundary value problem

$$x_{n+1} - f(x_n,\lambda) = 0, \quad n = n_-,\dots,n_+ - 1, \tag{4}$$

$$b(x_{n_-},x_{n_+},\lambda) = 0, \tag{5}$$

$$\Psi_{[n_-,n_+]}(x_{n_-},\dots,x_{n_+},\lambda) = 0. \tag{6}$$