SECOND ORDER AND BOOTSTRAP APPROXIMATION TO STUDENT'S t STATISTIC

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Abstract. We prove the validity of the one-term Edgeworth expansion for Student's t statistic under minimal conditions: the distribution of observations is non-lattice and has finite third moment. As a corollary we obtain the second-order correctness for the bootstrap of Student's t statistic. If the distribution satisfies Cramer's condition and has finite fourth moment the rate of the second order approximation is $O(N^{-1})$.

1. Introduction and Results

Let $X_1, \dots, X_N, \dots$ be independent identically distributed random variables. Write $\mathbf{E}\, X_1 = \mu$. Let

$$\mathbf{t} = \mathbf{t}(X_1, \dots, X_N) = (\overline{X} - \mu)/\hat{\sigma}$$

denote the Student statistic, where

$$\overline{X} = N^{-1}(X_1 + \cdots + X_N), \qquad \hat{\sigma}^2 = N^{-1} \sum\nolimits_{i=1}^{N} (X_i - \overline{X})^2.$$

Assume that $\sigma^2 := \mathbf{E}\ (X_1 - \mu)^2$ is finite and $\sigma^2 \neq 0$. Then the statistics $T_N = \sqrt{N}\mathbf{t}$ is asymptotically standard normal as $N \to \infty$.

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The accuracy of the normal approximation to the distribution function

$$F(x) = \mathbf{P}\{\sqrt{N}\mathbf{t} \le x\}$$

was studied by a number of authors, Helmers and van Zwet (1982), Helmers (1985), Slavova (1985), Hall (1988), Praskova (1989), Friedrich (1989), Bentkus and Götze (1996), Bentkus, Bloznelis and Götze (1996), etc. The Berry–Esseen bound $O(N^{-1/2})$ under the minimal moment condition $\mathbf{E}\,|X_1|^3 < \infty$ was obtained by Slavova (1985). Bentkus and Götze (1996) proved the bound $\sup_x |F(x) - \Phi(x)| < cN^{-1/2}\beta_3/\sigma^3$, thus extending the classical result of Esseen (1945) to the Studentized sums. Here $\Phi(x)$ denotes the standard normal distribution function and $\beta_3 := \mathbf{E}\,|X_1 - \mu|^3$.

A higher order approximation to the Student test was considered by Chung (1946), Bhattacharya and Ghosh (1978), Chibisov (1980), Babu and Singh (1985), Hall (1987), Putter and van Zwet (1997), Bentkus, Götze and van Zwet (1997), etc. Although increasingly general and precise, none of these results is optimal in the sense of expansions being established under minimal conditions. Hall (1987) proved the validity of a k-term Edgeworth expansion for Student's t statistic with remainder $o(N^{-k/2})$, for every integer k, provided that $\mathbf{E}\,|X_1|^{(k+2)/2} < \infty$ and the distribution F_0 of X_1 is non-singular. The moment conditions in Hall (1987) are the minimal ones, but the smoothness condition on the distribution F_0 of the observations is too restrictive. What is perhaps more important, Hall's result is only valid for a fixed underlying distribution function. As a result, it cannot be applied to the bootstrap.

The aim of the present paper is to prove the validity of an expansion under minimal conditions, in such a way that an extension to the bootstrap is straightforward. To this end, we approximate $F(x)$ by the one-term Edgeworth expansion

$$G(x) = \Phi(x) + \frac{\kappa_3}{6\sqrt{N}}\,(2\,x^2 + 1)\,\Phi'(x), \qquad \kappa_3 = \mathbf{E}\;X_1^3/\sigma^3$$

and construct bounds for the remainder

$$\Delta_N = \sup_x |F(x) - G(x)|.$$

The minimal smoothness condition which allows to prove the validity of one-term Edgeworth expansion, i.e., to prove the bound $\Delta_N = o(N^{-1/2})$, is the non-latticeness of the distribution F_0. We shall assume that F_0 is non-lattice.

Theorem 1. *Assume that $\mathbf{E}\,|X_1|^3 < \infty$ and the distribution of X_1 is nonlattice. Then*

$$\Delta_N = o(N^{-1/2}) \qquad as \quad N \to \infty. \tag{1.1}$$

Theorem 1 follows from Proposition 2.1, the formulation of which is rather technical and is deferred to Section 2.