THE DEFECT OF WEAK APPROXIMATION FOR HOMOGENEOUS SPACES

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Abstract. Let $X = H\backslash G$ be a homogeneous space defined over a number field k, where G is a connected linear k-group satisfying certain conditions, and H is a connected k-subgroup of G. For a finite set S of places of k we define a finite abelian group $A_S(X)$ which is the defect of weak approximation for X with respect to S. We describe $A_S(X)$ in terms of the Brauer group of X.

Introduction

Let k be an algebraic number field, $\mathcal{V}$ the set of places of k, and $S \subset \mathcal{V}$ a finite subset. Let X be an algebraic variety over k such that $X(k) \neq \varnothing$. We say that X *satisfies the condition of weak approximation with respect to* S if

$$(WA_S) \qquad X(k) \text{ is dense in } \prod_{v\in S} X(k_v)\,,$$

k_v denoting the completion of k at v. We say that X *satisfies the condition of weak approximation* if

$$(WA) \qquad X \text{ satisfies } (WA_S) \text{ for all finite } S\,.$$

This is the same as to require that $X(k)$ is dense in $\prod_{v\in\mathcal{V}} X(k_v)$.

Let $k_S = \prod_{v\in S} X(k_v)$; then $X(k_S) = \prod_{v\in S} X(k_v)$. Let $X(k)\widehat{_S}$ denote the closure of $X(k)$ in $X(k_S)$, and let $X(k)\widehat{_\mathcal{V}}$ denote the closure of $X(k)$ in $\prod_{v\in\mathcal{V}} X(k_v)$. Clearly, (WA_S) holds for X if and only if $X(k)\widehat{_S} = X(k_S)$, and (WA) holds for X if and only if $X(k)\widehat{_\mathcal{V}} = \prod_{v\in\mathcal{V}} X(k_v)$.

Let T be an algebraic k-torus. Set $A(T) = (\prod_{v\in\mathcal{V}} T(k_v))/T(k)\widehat{_\mathcal{V}}$. The abelian group $A(T)$ is the defect of weak approximation for T; in other words, it is the measure of failure of (WA) for T. In particular, T satisfies (WA) if and only if $A(T) = 0$. The group $A(T)$ was studied by Voskresenskiĭ [Vo1], [Vo2]. He showed that $A(T)$ is finite, and related it to a certain group $H^1(k, \mathrm{Pic}(\bar{V}(T)))$. As Sansuc later showed, $A(T)$ can be computed in terms of the Brauer group of T, cf. [Sa], 8.12.

For any connected linear k-group G one can define the set $A(G) = (\prod_{v\in\mathcal{V}} G(k_v))/G(k)\widehat{_\mathcal{V}}$. As above, $A(G)$ is the defect of weak approximation for

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G. Sansuc ([Sa], 3.3), generalizing Voskresenskiĭ's results, proved that the subgroup $G(k)\widehat{_\mathcal{V}}$ is normal and that the quotient group $A(G)$ is finite and abelian. He showed that it is possible to compute $A(G)$ in terms of the Brauer group of G, cf. [Sa], 8.12.

In this paper we consider the case of a homogeneous space. Let $X = H\backslash G$, where G is a connected linear k-group and $H \subset G$ is a connected k-subgroup. We suppose that $A(G) = 0$ and $Ш(G) = 0$, where $Ш(G)$ is the Tate-Shafarevich group of G. These conditions hold for example when G is a simply connected group, or an adjoint group, or an absolutely simple group, or if G splits over a cyclic extension, cf. [Sa], 5.3. For such X we construct a certain finite abelian group $A_S(X)$ which is the defect of (WA_S) for X. We construct $A_S(X)$ in terms of H and G, but then we compute $A_S(X)$ in terms of the Brauer group of X. We obtain a formula for $A_S(X)$ in terms of the Brauer group of X, similar to Sansuc's formula ([Sa], 8.12) for $A(G)$ in terms of the Brauer group of G for a connected k-group G.

The plan of the paper is as follows. In Section 1 we state the main results. In Section 2 we restate the problem of weak approximation for homogeneous spaces in terms of Galois cohomology. In Section 3 we prove a theorem describing $A_S(H\backslash G)$ in terms of H and G. In Section 4 we prove a theorem describing $A_S(H\backslash G)$ in terms of the Brauer group of $H\backslash G$.

The results of this paper were announced in [Bo1]. The case of a homogeneous space $H\backslash G$ of a *simply connected* group G was treated in [Bo2].

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The paper is inspired by Sansuc's paper [Sa]. The computation of $A_S(H\backslash G)$ in terms of H and G is based on results of Kottwitz [Ko1], [Ko2] in the more functorial version of [Bo3].

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Notation

k is a field of characteristic 0, $\bar{k}$ is an algebraic closure of k, $\Gamma = \mathrm{Gal}(\bar{k}/k)$. When k is a number field, then $\mathcal{V}$, $\mathcal{V}_\infty$, $\mathcal{V}_f$ are the set of all places of k, of infinite places, of finite places, resp. For $v \in \mathcal{V}$, k_v denotes the completion of k at v. For a finite subset $S \subset \mathcal{V}$ we set $k_S = \prod_{v\in S} k_v$. We write $\mathbf{A}$ for the adèle ring of k. For a connected k-group G we set

$$H^1(k_S, G) = \prod_{v\in S} H^1(k_v, G), \quad H^1(\mathbf{A}, G) = \bigoplus_{v\in\mathcal{V}} H^1(k_v, G)\,,$$

where $\bigoplus$ denotes the subset of the direct product consisting of families (ξ_v) for which $\xi_v = 1$ for all but a finite number of places v.

Let G be a connected linear group over a field F. Then:
G^{u} is the unipotent radical of G.
$G^{\mathrm{red}} = G/G^{\mathrm{u}}$; it is a reductive group.
$G^{\mathrm{ss}} = (G^{\mathrm{red}})^{\mathrm{derived}}$; it is a semisimple group.
$G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$, it is a torus.