SCATTERING PROBLEM FOR LOCAL PERTURBATION OF FREE QUANTUM GAS.

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Abstract. Scattering theory for perturbations of the intrinsic Dirichlet (Laplace-Beltrami) operator $H_0 = -\operatorname{div}^\Gamma \nabla^\Gamma$ on $L_2(\Gamma, \pi_z)$, *ie* the space of π_z-square integrable functions on the configuration space Γ over $\mathbb{R}^d$, is studied. Here π_z denotes Poisson measure with intensity z. We show that for an arbitrary regular non-zero potential V the standard wave operators $W^\pm(H_0, H_0 + V)$ do not exist, and propose to consider Dirichlet operators of perturbed Poisson measures instead of potential perturbations of the Hamiltonian H_0. As case studies, cylindric smooth densities and finite volume Gibbs perturbations of the Poisson measure are considered. In these cases the existence of the corresponding wave operators is proved.

KEY WORDS: Scattering theory, Poisson analysis, Dirichlet forms.
RUNNING HEAD: Scattering problem for quantum gas

Introduction

Finite as well as zero temperature states of free quantum gases are usually described in terms of the Araki–Woods representation (see, *eg*, [5]). In particular, the free Bose gas of zero density has a standard representation in Fock space, which leads to the known Schrödinger representation in the functional realization of the Fock space as a space of functions square integrable with respect to the corresponding Gaussian measure, see *eg*, [4]. In contrast to this, the free Bose gas of positive density cannot be properly described in the Gaussian space representation, and a rigorous mathematical description of the ground state for this system can only be achieved in terms of Poisson analysis (see [3] as well as the references and historical comments therein).

In this paper we study scattering theory for different types of perturbations of the intrinsic Dirichlet (Laplace–Beltrami) operator $H_0 = -\operatorname{div}^\Gamma \nabla^\Gamma$ on $L_2(\Gamma, \pi_z)$ [1], where π_z is the Poisson measure with Lebesgue intensity measure $z dx$, $z > 0$, and Γ is the corresponding configuration space over $\mathbb{R}^d$. It should be mentioned here that from the physical point of view the operator H_0 is the energy operator, or Hamiltonian, of the corresponding system of the free Bose gas [3]. In Section 2 we show that for an arbitrary regular non-zero potential perturbation V the standard wave operators $W^\pm(H_0, H_0 + V)$ do not exist. (An analogous result in the Gaussian case is well known, see *eg* [4].) Hence, to study scattering theory for the pair of operators H_0 and $H_0 + V$, we need some renormalization of the perturbed Hamiltonian $H_0 + V$. As in the case of Gaussian analysis [4], such a renormalization can be given by the Dirichlet operator of a perturbed Poisson measure (see [2, 3]). In Section 3 we consider perturbations of π_z by cylindric smooth densities and prove existence of the corresponding wave operators. Most important from the mathematical physics point of view is the case of a local perturbation of the Poisson measure by a Gibbs factor, describing a locally perturbed Bose gas in which interactions between particles take place only in a fixed bounded domain of $\mathbb{R}^d$.

But first, we need to recall the notions of configuration space, Poisson measure and the corresponding Dirichlet form [1, 2] (Section 1).

1. Poisson analysis and intrinsic differential geometry of Poisson spaces

In this section we provide a brief review of Poisson analysis and the intrinsic differential geometry of the corresponding Poisson spaces needed for this article. For a more detailed exposition of different aspects of Poisson

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analysis and the intrinsic differential geometry of Poisson spaces as well as their applications to quantum field theory and statistical mechanics see [7, 8, 1, 2, 3, 18, 20, 13, 14, 12, 16] and the references therein.

Let $\mathcal{L}(\Lambda)$, $\Lambda \subseteq \mathbb{R}^d$, denote the system of all bounded Borel subsets in Λ and define the configuration space Γ over $\mathbb{R}^d$ as the set of all locally finite configurations in $\mathbb{R}^d$, *ie*

$$\Gamma := \{\gamma \subset \mathbb{R}^d \mid \#(\gamma \cap \Lambda) < \infty \text{ for all } \Lambda \in \mathcal{L}(\mathbb{R}^d)\},$$

$\#(\gamma \cap \Lambda)$ being the cardinality of the set $\gamma \cap \Lambda$.

Let $\mathfrak{B}_\Lambda(\Gamma)$, $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, be the σ-algebra on Γ generated by all mappings of the form

$$\Gamma \ni \gamma \mapsto \#(\gamma \cap \Lambda') \in \mathbb{Z}_+ := \{0, 1, 2, \dots\},$$

where $\Lambda' \in \mathcal{L}(\Lambda)$. We set $\mathfrak{B}(\Gamma) := \mathfrak{B}_{\mathbb{R}^d}(\Gamma)$.

The Poisson measure $\pi_{\tilde{z}}$ over $\mathfrak{B}(\Gamma)$ is defined so that for all $\Lambda \in \mathcal{L}(\mathbb{R}^d)$ and any $\mathfrak{B}_\Lambda(\Gamma)$-measurable function $F \geq 0$

$$\int_\Gamma F(\gamma)\pi_{\tilde{z}}(d\gamma) = \exp\Big[-\int_\Lambda \tilde{z}(x)dx\Big] \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} F\Big(\{x_i\}_{i=1}^n\Big)(\tilde{z}dx)_n^1, \tag{1.1}$$

where $(\tilde{z}dx)_n^1 := \tilde{z}(x_1)\dots\tilde{z}(x_n)\,dx_1\dots dx_n$ and $\tilde{z}$ is some non-negative function from $L_1^{\mathrm{loc}}(\mathbb{R}^d)$. The existence of $\pi_{\tilde{z}}$ is insured by Kolmogorov's theorem.

Let us set for an arbitrary $\mathfrak{B}(\mathbb{R}^d)$-measurable function f over $\mathbb{R}^d$ with bounded essential support

$$\langle f, \gamma\rangle := \sum_{x\in\gamma} f(x).$$

This means that we identify a configuration $\gamma \in \Gamma$ with the following point measure over $\mathfrak{B}(\mathbb{R}^d)$:

$$\gamma \equiv \sum_{x\in\gamma} \delta_x,$$

where δ_x is Dirac-measure at the point x, *ie*, for all $A \in \mathfrak{B}(\mathbb{R}^d)$

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1. *Let $\xi \in S(\mathbb{R}^d)$, $\Lambda \in \mathcal{L}(\mathbb{R}^d)$, and let $F : \Gamma \mapsto \mathbb{R}$ be a $\mathfrak{B}_\Lambda(\Gamma)$-measurable function and $\tilde{z} \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$. Then*

$$\int_\Gamma e^{\langle\xi,\gamma\rangle} F(\gamma)\pi_{\tilde{z}}(d\gamma) = \exp\Big[\int_{\mathbb{R}^d} (e^{\xi(x)} - 1)\tilde{z}(x)dx\Big] \int_\Gamma F(\gamma)\pi_{\tilde{z}e^\xi}(d\gamma), \tag{1.2}$$

provided the integrals with respect to $\tilde{z}dx$ and the Poisson measure $\pi_{\tilde{z}e^\xi}$ in the right-hand side of (1.2) exist.

Proof. Since $\mathcal{D}(\mathbb{R}^d) := C_0^\infty(\mathbb{R}^d)$ is dense in $S(\mathbb{R}^d)$, it is enough to prove (1.2) for $\xi \in \mathcal{D}(\mathbb{R}^d)$. We may assume that $\Lambda \supset \operatorname{supp}\xi$. Then, using formula (1.1), we obtain:

$$\begin{aligned}\int_\Gamma e^{\langle\xi,\gamma\rangle} F(\gamma)\pi_{\tilde{z}}(d\gamma) &= \exp\Big[-\int_\Lambda \tilde{z}(x)dx\Big] \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} e^{\xi(x_1)+\dots+\xi(x_n)} F\Big(\{x_i\}_{i=1}^n\Big)(\tilde{z}\,dx)_n^1 = \\ &= \exp\Big[\int_\Lambda (e^{\xi(x)} - 1)\tilde{z}(x)dx\Big] \exp\Big[-\int_\Lambda \tilde{z}(x)e^{\xi(x)}dx\Big] \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} F\Big(\{x_i\}_{i=1}^n\Big)(\tilde{z}e^\xi\,dx)_n^1 = \\ &= \exp\Big[\int_{\mathbb{R}^d} (e^{\xi(x)} - 1)\tilde{z}(x)dx\Big] \int_\Gamma F(\gamma)\pi_{\tilde{z}e^\xi}(d\gamma). \qquad \square\end{aligned}$$

For an arbitrary measurable function f_n over $\mathbb{R}^{dn}$ with bounded essential support and $\gamma \in \Gamma$ we define a Wick monomial $\langle f_n, :\gamma^{\otimes n}:\rangle$ as a function of $\gamma \in \Gamma$ by the following formula [12]

$$\langle f_n, :\gamma^{\otimes n}:\rangle := \sum_{\{x_1,\dots,x_n\}\subset\gamma} \sum_{\pi\in\mathfrak{S}_n} f_n(x_{\pi(1)}, \dots, x_{\pi(n)}), \tag{1.3}$$

where $\mathfrak{S}_n$ denotes the group of all permutations of the set $\{1, \dots, n\}$.