HODGE DECOMPOSITION FOR HIGHER ORDER HOCHSCHILD HOMOLOGY

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0. Introduction

Let A be a commutative algebra and M be an A-module. Let

$$\mathcal{L}(A,M):\Gamma\to Vect$$

be a functor from finite pointed sets to the category of vector spaces given by

$$\{0,1,...,n\}\mapsto M\otimes A^{\otimes n},$$

see also section 1.7. For any based simplicial finite set $Y:\Delta^{op}\to\Gamma$ we let $H_*^Y(A,M)$ denote the homology of the chain complex associated to the simplicial vector space

$$\Delta^{op}\xrightarrow{Y}\Gamma\xrightarrow{\mathcal{L}(A,M)}Vect.$$

For $Y=S^d, d\geq 1$, one uses the notion $H_*^{[d]}(A,M)$ instead of $H_*^{S^d}(A,M)$. For $Y=S^1$ one recovers the usual Hochschild homology of A with coefficient in M (see [L2]), because of this fact, we call the groups $H_*^{[d]}(A,M)$ as *Hochschild homology of order* d of A with coefficient in M. Higher order Hochschild homology was implicitly defined in [A1]. These groups are related to the cohomology of the mapping spaces X^{S^d} in the same way as usual Hochschild homology is related to the cohomology of free loop space X^{S^1} (see section 5). The main goal of this paper is to show that the higher order Hochschild homology in charecteristic zero case have naturaral decomposition, which for $d=1$ is isomorphic to the classical Hodge decomposition ([L2]). Our methods are new even for $d=1$ and are based on homological properties of Γ-modules.

Let us recall that the Hochschild and cyclic homology of commutative algebras in characteristic zero case have natural decompositions, known as Hodge decomposition. This result for Hochschild homology was proved by Quillen ([Q]). Using sophisticated combinatorics Gerstenhaber and Schack constructed the so called Eulerian idempotents and proved that these yield a decomposition of Hochschild homology (see [GS], [L1]). It turns out that both decompositions are isomorphic to each other (see [Ro]). Moreover, Loday made in [L1] the important observation that such decompositions have a more general nature. Namely he proved the following result: Let $F : \Gamma \to Vect$ be a functor from finite pointed sets to the category of vector spaces over a characteristic zero field. Let $S^1 : \Delta^{op} \to Sets_*$ be the standard simplicial model for the circle, which has only two nondegenerate simplices. Clearly the values of S^1 are in finite pointed sets, so one can take the composition of these two functors $F(S^1) : \Delta^{op} \to Vect$. In this way one gets the simplicial vector space $F(S^1)$ and one can take the homotopy groups of it. Among other things, Loday proved that there exists a natural decomposition

$$\pi_n F(S^1) \cong \bigoplus_{i=0}^{n} H_n^{(i)}(F), \ n \geq 0,$$

where the groups $H_n^{(i)}(F)$ are defined using Eulerian idempotents. If A is a commutative algebra and M is a symmetric bimodule, then for $F = \mathcal{L}(A, M)$ Loday's result gives exactly the Hodge decomposition for Hochschild homology. Loday obtained in [L1] also the similar decomposition for cyclic homology (see section 3 for more details). Later McCarthy remarked that Loday's decomposition can be obtained using the rotations of a circle [M]. We refer to [W] for relationship between Hodge decomposition of cyclic homology and Hodge filtration in algebraic geometry and [BV] and [V] for related results.

We give an alternative approach of this subject. We do not only give a new purely homological proof of the results of Loday, but we obtain essentially more. Using homological algebra of Γ-modules we constract a spectral sequence, whose abutement is $\pi_* F(L)$. Here L is any based simplicial finite set. We show that for $L = S^d$, the spectral sequence stops at E^2 level and gives a natural decomposition for $\pi_n F(L)$. We give a simple axiomatic characterization of the decomposition of $\pi_* F(S^1)$, from which we deduce that our and Loday's decomposition are isomorphic. The striking fact is that in