On a Variational Formulation of QSVD and RSVD

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Abstract

Recently, Chu, Funderlic and Golub [SIAM J. Matrix Anal. Appl., 18:1082–1092, 1997] presented a variational formulation for the quotient singular value decomposition (QSVD) of two matrices $A \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{p\times m}$ which is a generalization of that one for the ordinary singular value decomposition (OSVD) and characterizes the role of two orthogonal matrices in QSVD. In this paper, we give an alternative derivation of this variational formulation and extend it to establish an analogous variational formulation for the Restricted Singular Value Decomposition (RSVD) of Matrix Triplets $A \in \mathbf{R}^{n\times m}, B \in \mathbf{R}^{n\times l}, C \in \mathbf{R}^{p\times m}$ which provides new understanding of the orthogonal matrices appearing in this decomposition.

Keywords: OSVD, QSVD, RSVD, Generalized Singular Value, Variational Formulation, Stationary Value, Stationary Point.
AMS subject classification: 65F15, 65H15.

1 Introduction

The ordinary singular value decomposition (OSVD) of a given matrix $A \in \mathbf{R}^{n\times m}$ is

$$U^TAV = \begin{array}{c} \\ r_a \\ n-r_a \end{array} \begin{array}{c} \begin{array}{cc} r_a & m-r_a \end{array} \\ \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \tag{1}$$

with

$$\begin{aligned} U &= n \begin{array}{c} \begin{array}{cc} r_a & n-r_a \end{array} \\ \begin{bmatrix} U_1 & U_2 \end{bmatrix} \end{array}, \quad V = m \begin{array}{c} \begin{array}{cc} r_a & m-r_a \end{array} \\ \begin{bmatrix} V_1 & V_2 \end{bmatrix} \end{array}, \\ \Sigma &= \operatorname{diag}\{\sigma_1, \cdots, \sigma_{r_a}\}, \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{r_a} > 0, \qquad r_a = \operatorname{rank}(A), \end{aligned}$$

where U, V are orthogonal matrices. The $\sigma_1, \cdots, \sigma_{r_a}$ are the non-trivial singular values of A, and the columns of U_1 and V_1 are, respectively, the non-trivial left and right singular vectors of A. In this paper, $\|\cdot\|$ denotes the $2-$norm of a vector. The following theorem is well-known [4]:

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Theorem 1 *Given $A \in \mathbf{R}^{n\times m}$ with OSVD (1).*

(a) Consider the optimization problem

$$\max_{y=Ax,\ y\neq 0} \frac{\|y\|}{\|x\|}. \tag{2}$$

Then the non-trivial singular values $\sigma_1, \cdots, \sigma_{r_a}$ of A are precisely the stationary values, i.e., the functional evaluations at the stationary points, of (2). And, let the stationary points in (2) corresponding to the stationary values $\sigma_1, \cdots, \sigma_{r_a}$ be $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \cdots, \begin{bmatrix} x_{r_a} \\ y_{r_a} \end{bmatrix}$, then

$$V_1 = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \cdots & \frac{x_{r_a}}{\|x_{r_a}\|} \end{bmatrix}.$$

Moreover, if $n = m = r_a$, then

$$U_1 = \begin{bmatrix} \frac{y_1}{\|y_1\|} & \cdots & \frac{y_{r_a}}{\|y_{r_a}\|} \end{bmatrix}.$$

(b) Consider the dual optimization problem

$$\max_{y^T=x^TA,\ y\neq 0} \frac{\|y\|}{\|x\|}. \tag{3}$$

Then the non-trivial singular values $\sigma_1, \cdots, \sigma_{r_a}$ of A are precisely the stationary values of (3). And, let the stationary points in (3) corresponding to the stationary values $\sigma_1, \cdots, \sigma_{r_a}$ be $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \cdots, \begin{bmatrix} x_{r_a} \\ y_{r_a} \end{bmatrix}$, then

$$U_1 = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \cdots & \frac{x_{r_a}}{\|x_{r_a}\|} \end{bmatrix}.$$

Moreover, if $n = m = r_a$, then

$$V_1 = \begin{bmatrix} \frac{y_1}{\|y_1\|} & \cdots & \frac{y_{r_a}}{\|y_{r_a}\|} \end{bmatrix}.$$

Recently, in Chu, Funderlic and Golub [1] Theorem 1 has been generalized to the Quotient Singular Value Decomposition (QSVD) [3, 5, 6, 7, 8, 9, 10, 11, 13, 14] of two matrices $A \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{p\times m}$ based on the relationship between QSVD of two matrix A, C and the eigendecomposition of the matrix pencil (A^TA, C^TC).

The purposes of this paper are twofold. Firstly, we present an alternative derivation of the variational formulation in [1] directly based on the QSVD of two matrices A, C. Then we extend this result to the Restricted Singular Value Decomposition (RSVD)[8, 9, 10, 11, 15, 16] of matrix triplets and obtain a analogous variational formulation which provides new understanding of the orthogonal matrices appearing in this decomposition.

In order to prove our main results, we will establish two condensed forms based on orthogonal matrix transformations. The QSVD of two matrices and the RSVD of matrix triplets can be obtained and the variational formulation for QSVD and RSVD can be proved directly based on these two condensed forms.

In this paper, we use the following notation:

- $\mathcal{S}_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix M;