ALTERNATING QUOTIENTS OF FUCHSIAN GROUPS

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Abstract. It is shown that any Fuchsian group has among its homomorphic images all but finitely many of the alternating groups A_n. This settles in the affirmative a long-standing conjecture of Graham Higman.

1. Introduction

It all started with a theorem of G. A. Miller. Suppose X is the 2-sphere S^2, the Euclidean plane $\mathbb{E}^2$ or the hyperbolic plane $\mathbb{H}^2$. Let G be a discrete group of orientation preserving isometries of X. By classical work of Fricke and Klein (see for instance [22]), G, if finitely generated and nonelementary, has a presentation,

generators:	$a_1, b_1, \dots, a_g, b_g$	(hyperbolic)
	$x_1, \dots, x_e$	(elliptic)
	$y_1, \dots, y_s$	(parabolic)
	$z_1, \dots, z_t$	(hyperbolic boundary elements)
relations:	$x_1^{m_1} = \cdots = x_e^{m_e} = 1,$	
	$\prod_{i=1}^{e} x_i \prod_{j=1}^{s} y_j \prod_{k=1}^{t} z_k \prod_{l=1}^{g} [a_l, b_l] = 1.$	

When $X = \mathbb{H}^2$, G is called a *Fuchsian* group.

The division into spherical, Euclidean and Fuchsian is governed by the quantity,

$$\mu(G) = 2g - 2 + \sum_{i=1}^{e} \left(1 - \frac{1}{m_i}\right) + s + t, \tag{1}$$

where $\mu(G) < 0, = 0$ or > 0 as $X = S^2, \mathbb{E}^2$ or $\mathbb{H}^2$. The quotient X/G is an orientable 2-orbifold of genus g with e cone points, s punctures and t boundary components. Its geometry and the algebraic structure of G are intimately connected. As a consequence, G is determined upto isomorphism by the parameters $(g; m_1, \dots, m_e; s; t)$, $2 \leq m_1 \leq \cdots \leq m_e$, its *signature*.

The spherical and Euclidean groups have an algebraic structure that is fairly transparent; the former are finite and the latter are finite abelian extensions of $\mathbb{Z} \oplus \mathbb{Z}$, hence soluble. Fuchsian groups are a different matter altogether. To get a firmer grip on this situation, one may be tempted to consider finite (even simple) homomorphic images of these groups. The first appearance of such in the literature would seem to be Miller's result [15] on the group of signature $(0; 2, 3; 1; 0)$–the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$. Namely, one can find among its homomorphic

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images every alternating group, bar A_6, A_7 and A_8. There is an added geometric incentive to such results: any compact Riemann surface (or complex algebraic curve) has as its group of conformal automorphisms a finite homomorphic image of a Fuchsian group.

In the late 1960's, Graham Higman conjectured that *any* Fuchsian group has among its homomorphic images all but finitely many of the alternating groups. In [3, 4, 7, 16, 17], the case where G has signature $(0; 2, m_1, m_2; 0; 0)$ was considered and essentially settled in the affirmative. This paper considerably extends this body of work to finally settle Higman's conjecture. So, taking the phrase "essentially all" to be synonymous with "all but finitely many", and "surjects" with "has among its homomorphic images", we prove

Main Theorem. Any Fuchsian group surjects essentially all of the alternating groups.

The theorem has a 3-dimensional interpretation also. For, suppose M is a 3-dimensional Seifert fibred space, that is, an S^1-bundle over a 2-orbifold (see [9, 14]). Since the fundamental group of M surjects π_1(underlying 2-orbifold), we have

Corollary 1. If M is Seifert fibred over an orientable hyperbolic 2-orbifold, then $\pi_1(M)$ surjects essentially all of the alternating groups.

The machinery used to prove the theorem uses what are essentially Schreier coset diagrams as a vehicle. Such diagrams appear in the literature in many guises–they are equivalent to *hypermaps* (see [11]) in the case $G = (0; m_1, m_2, m_3; 0; 0)$, and *dessin d'enfants*, as Grothendieck would have it, when $G = (0; 2, m_2, m_3; 0; 0)$–see [2].

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2. The plan

The proof of the theorem proceeds according to various cases. The result of this sections musings will be that the general result reduces to an investigation of the cocompact groups (that is, $s = t = 0$) of genus zero, or *Dyck groups*. The most important, and the hardest of these in turn, will be those of signature $(0; m_1, m_2, m_3; 0; 0)$–the *triangle groups*.

From the presentation, a group of signature $(g; m_1, \dots, m_e; s; t)$ is isomorphic to one of signature $(g; m_1, \dots, m_e; s + t; 0)$. Moreover by (1), the former is Fuchsian if and only if the latter is. We may assume then, without loss of generality, that $t = 0$. Write $(g; m_1, \dots, m_e; s)$ for $(g; m_1, \dots, m_e; s; 0)$ from now on.

We can surject a group with signature $(g; m_1, \dots, m_e; s)$ onto one of signature $(g'; m_1, \dots, m_i', \dots, \hat{m}_j, \dots, m_e; s')$, where $g' \leq g$, $s' \leq s$, m_i' is a divisor of m_i, and the hat denotes ommission. Here's how: map all but g' of the a_i's, and the corresponding b_i's to the identity; send x_i to $x_i^{m_i/m_i'}$, x_j to the identity and all but s' of the parabolics to the identity. All other generators map to themselves. The map extends to the desired homomorphism.