On a Cone-Manifold with the Euclidean Structure on the Whitehead Link *

Ruslan Shmatkov †

Abstract

A 3-dimensional cone-manifold $S^3(W, 2\pi/\alpha)$ with the Euclidean structure on the Whitehead link is studied in this article. A method is given to construct the canonical fundamental set $FS(\alpha)$ of the cone-manifold $S^3(W, 2\pi/\alpha)$ in the 3-dimensional Euclidean space $\mathbb{E}^3$. The desired fundamental set $FS(\alpha)$ is obtained as a 32-faced polyhedron in $\mathbb{E}^3$. This fundamental set is the simplest one in some sense.

1 Introduction

The investigation of cone-manifolds was started in [1, 2]. Then vital exploration of these manifolds was continued in [3, 4, 5, 6, 7, 8]. One of the fundamental results of this theory is the Rigidity Theorem, which is recently proved by S. Kojima.

This work was inspired by papers [7] and [9].

In the paper [7] H. Helling, A.C. Kim, J.L. Mennicke for the first time strictly proved that the orbifold $S^3(W, 2\pi/n)$ is hiperbolic for all $n = 3, 4, 5, \ldots$.

Then, in the paper [9] A. Mednykh and A. Rasskazov constructed the canonical fundamental set for 2-bridge link orbifolds.

In this paper some 3-dimensional cone-manifold $S^3(W, 2\pi/\alpha)$, where $\alpha = 2.506179\ldots$, with the Euclidean structure on the Whitehead link is studied. This 3-manifold belongs to an infinite family of cone-manifolds $S^3(W, \beta)$ with the 3-dimensional sphere $\mathbb{S}^3$ as the underlying space, the Whitehead link W as the singular set, and the angle β, $0 < \beta < 2\pi$. Using results of [3, 5] it is possible to show that $S^3(W, \beta)$ is hyperbolic if $0 < \beta < 2\pi/\alpha$ and spherical if $\beta > 2\pi/\alpha$.

Following K.N.Jones [8] we say that a *3-dimensional Euclidean cone-manifold* $X^3(\Sigma, \beta)$ is a metric space obtained as the quotient space of a disjoint union of a collection of geodesic 3-simplices in $\mathbb{E}^3$ by an isometric pairing of codimension-one faces in such a combinatorial fashion that the underlying topological spase is a manifold. *Hyperbolic* and *spherical cone-manifolds* are defined similarly.

Such a space possesses a Riemannian metric of constant sectional curvature on the union of the top-dimensional cells and the codimension-1 cells. On each codimension-2 cell, the structure is completely described by an angle β, which is the sum of the dihedral angles around all of the codimension-2 simplicial faces which are identified to give the cell. The *singular set* of a cone-manifold is the closure of all the codimension-2 cells for which this angle β, called the *cone angle* , is not 2π (the Riemannian metric may be extended smoothly over all cells whose

*Supported by the RFFI (grant 96-01-01523)
†Sobolev Institute of Mathematics, Novosibirsk-90, 630090, RUSSIA, e-mail: ruslan@math.nsc.ru

angle is 2π). In this paper, we consider the case where the singular set is a link with a constant cone angle $0 < \beta < 2\pi$ on each component.

We should note here that there is a strong connection between cone-manifolds and orbifolds (see [2]), namely that *orbifolds* are cone-manifols with all cone angles of the form $2\pi/k$ for some integer k (k is the order of the isotropy of a non-vertex cone point).

Denote by $S^3(W, 2\pi/\alpha)$ the cone-manifold $X^3(\Sigma, \beta)$, whose underlying space X^3 is $\mathbb{S}^3$, the singular set Σ is homeomorphic to the Whitehead link W, and the cone angle $\beta = 2\pi/\alpha$.

The aim of this paper is to prove the following theorem.

Theorem 2.1. *Let $0 < \beta < 2\pi$ be the real root of the equation:*

$$9 + 8\cos\beta - 13\cos 2\beta + 4\cos 3\beta = 0$$

and $\alpha = 2\pi/\beta$.

Then there exists a 32-faced polyhedron in $\mathbb{E}^3$ which is the canonical fundamental set of the cone-manifold $S^3(W, 2\pi/\alpha)$. In particular, $S^3(W, 2\pi/\alpha)$ admits the Euclidean structure.

This paper is organized as follows.

In Section 1, the standard Wirtinger procedure [10] yields a presentation for the fundamental group $\pi_1(\mathbb{S}^3 \setminus W)$ of the complement of the Whitehead link W.

Let Γ be the presentation for the fundamental group $\pi_1(\mathbb{S}^3 \setminus W)$ in the group $SU(2)$, which satisfies the following conditions on generators of Γ (denoted by S and T):

$$r(S,T) = T^{-1}S^{-1}TS^{-1}T^{-1}STS^{-1}TST^{-1}STS^{-1}T^{-1}S = I. \tag{1}$$

$$\mathrm{tr}(S) = \mathrm{tr}(T) = 2\cos\frac{\pi}{\alpha}, \tag{2}$$

where tr(S) denotes the trace of the matrix S.

Using (1), we obtain the following equation:

$$9 + 8\cos\beta - 13\cos 2\beta + 4\cos 3\beta \quad = \quad 0, \tag{3}$$

where $\beta = 2\pi/\alpha$. Studing this equation we obtain that it has exactly one real root $0 < \beta < 2\pi$ and the number $\alpha = 2\pi/\beta$ is irrational. Since α is irrational, Γ is not discrete. However we can use Γ to construct the canonical fundamental set $FS(\alpha)$ of the cone-manifold $S^3(W, 2\pi/\alpha)$.

In Section 2 the canonical fundamental set $FS(\alpha)$ of the cone-manifold $S^3(W, 2\pi/\alpha)$ in the 3-dimensional Euclidean space $\mathbb{E}^3$ is constructed.

For this purpose we find images of S and T in the group $SO(3)$, using the canonical epimorphism $SU(2) \to SO(3)$.

The method proposed allows to derive explicit formulas for coordinates of vertices of the fundamental set $FS(\alpha)$ in $\mathbb{E}^3$.

Note, that $FS(\alpha)$ is the simplest possible fundamental set for $S^3(W, 2\pi/\alpha)$ in the sense that it has just two pairs of mutually congruent curvelinear faces, which are pairwise identified by rotations of the Euclidean space $\mathbb{E}^3$.

Similar construction for the figure-eight knot was realized in [9]. This method can be used for investigation of Euclidean structures on arbitrary 2-bridge knotes and links.