ON THE VALIDITY OF THE LOG–SOBOLEV INEQUALITY FOR SYMMETRIC FLEMING–VIOT OPERATORS

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Abstract. We prove that Fleming–Viot operators with parent–independent mutation satisfy a logarithmic Sobolev inequality if and only if the set of types is finite.

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1. Introduction

Let (X, μ) be a finite measure space and $(\mathcal{E}, D(\mathcal{E}))$ be a densely defined (not necessarily closed) quadratic form on $L^2(\mu)$. $(\mathcal{E}, D(\mathcal{E}))$ is said to determine a logarithmic Sobolev inequality with constant $c > 0$ if

$$\int f^2 \log f^2 \, d\mu \leq c\,\mathcal{E}(f,f) + \|f\|^2_{L^2(\mu)} \log \|f\|^2_{L^2(\mu)}$$

for all $f \in D(\mathcal{E})$. Such kind of inequality has been invented in the context of quantum field theory as a tool to prove hypercontractivity of semigroups associated with certain infinite–dimensional elliptic differential operators. Meanwhile, this tool has found many other applications also in finite dimensions and the logarithmic Sobolev inequality has been verified in the case of many important examples of stochastic analysis (cf. [Ba], [Gr] and references therein). Hence it is a remarkable fact that in the class of measure–valued diffusions there is up to now not one single example in which a logarithmic Sobolev inequality has been verified. The purpose of this paper now is to give an answer to the question, whether or not a logarithmic Sobolev inequality holds for generators of Fleming–Viot processes with parent–independent mutation and, if a logarithmic Sobolev inequality does not hold, whether or not we can find a reasonable substitute for this inequality. Fleming–Viot processes can

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be viewed as diffusion approximations of empirical processes associated with a certain class of discrete time Markov chains in population genetics (cf. [EK2]) and are (apart from Dawson–Watanabe processes) the best studied class of measure–valued diffusions. Before we state our main result let us first define Fleming–Viot processes: Let S be a complete separable metric space which is interpreted as a space of types of a given population. Throughout this paper we will assume that S is compact. Let $E := \mathcal{M}_1(S)$ be the space of all probability measures on S (i.e., all possible distributions of types within the given population) equipped with the weak topology. One can then introduce random mutation on the population with the help of a Feller generator A (i.e., the generator of a sub–Markovian C_0–semigroup on the space $C(S)$ of all continuous functions on S). Throughout the whole paper we will only consider bounded mutation operators of the following type

$$Af(x) = \frac{\theta}{2} \int_S (f(y) - f(x))\, \nu_0(dy)\,, f \in C(S)\ ,$$

where $\theta > 0$ and $\nu_0 \in E$ such that $\mathrm{supp}\,(\nu_0) = S$. The Fleming–Viot process associated with mutation operator A (with no recombination and no selection) is called the Fleming–Viot process with parent–independent mutation and is defined as the unique solution of the $C_E[0,\infty)$–martingale problem $(L_{\theta,\nu_0}, \mathcal{F}C^\infty)$, where

$$\mathcal{F}C^\infty := \{F = \varphi(\langle f_1, \cdot\rangle, \dots, \langle f_d, \cdot\rangle) | f_i \in C(S), \varphi \in C^\infty(\mathbb{R}^d), d \in \mathbb{N}\} \tag{1.1}$$

and

$$\begin{aligned} L_{\theta,\nu_0}F(\mu) := {} & \frac{1}{2} \sum_{i,j=1}^{d} (\partial_{x_i}\partial_{x_j}\varphi)(\langle f_1, \mu\rangle, \dots, \langle f_d, \mu\rangle) \mathrm{cov}_\mu(f_i, f_j) \\ & + \frac{1}{2} \sum_{i=1}^{d} (\partial_{x_i}\varphi)(\langle f_1, \mu\rangle, \dots, \langle f_d, \mu\rangle) \langle Af_i, \mu\rangle \end{aligned} \tag{1.2}$$

(cf. [EK2]). Here $\langle f, \mu\rangle := \int f\, d\mu$. It is well–known that this process has a unique stationary distribution $m_{\theta,\nu_0} \in \mathcal{M}_1(E)$ which is even symmetrizing (cf. [EK2, 8.1]). m_{θ,ν_0} can be described as follows: Let $(\rho_1, \rho_2, \dots)$ have a Poisson–Dirichlet distribution with parameter θ and let $(\xi_n)_{n\in\mathbb{N}}$ be i.i.d. with distribution ν_0 and independent of $(\rho_1, \rho_2, \dots)$. Then

$$m_{\theta,\nu_0}[A] = P\left[\sum_{i=1}^{\infty} \rho_i \delta_{\xi_i} \in A\right] . \tag{1.3}$$

The Dirichlet form $(\mathcal{E}_{\theta,\nu_0}, H^{1,2}(m_{\theta,\nu_0}))$ associated with the symmetric Fleming–Viot operator $(L_{\theta,\nu_0}, \mathcal{F}C^\infty)$ is obtained as the closure of the bilinear form $(-L_{\theta,\nu_0}F, G)_{L^2(m_{\theta,\nu_0})}$, $F, G \in \mathcal{F}C^\infty$, in $L^2(m_{\theta,\nu_0})$. We will prove in Proposition 3.4 that $\mathcal{E}_{\theta,\nu_0}$ determines a Poincaré inequality with constant $\frac{2}{\theta}$ (i.e., the corresponding generator has a mass gap of size $\frac{\theta}{2}$). We will show in the appendix that the existence of a mass gap can also be deduced from a result obtained by S.N.