# REPRESENTATIONS OF BURNSIDE RINGS

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Abstract. Motivated from equivariant topology we investigate representations of Burnside rings, i.e. modules which are finitely generated and free as abelian groups. We prove a necessary and sufficient condition for a Burnside ring to be of finite representation type. For $p$-groups whose Burnside ring is of finite representation type all the indecomposable representations are classified. Finally, we discuss a class of Burnside rings whose representation theory is particularly nice.

## Introduction

Although the objects of our discussion are of some interest by themselves, the main motivation for studying Burnside rings and their modules originates from their natural occurrence in equivariant topology. Before summarizing the purely algebraic content of this article, we roughly sketch this topological background. More details can be found in [8].

Let $G$ be a finite group. Denote by $S^V$ the one point compactification of a complex $G$-representation $V$ with base point at infinity. The equivariant stable homotopy group of pointed $G$-CW-complexes, say $X$ and $Y$, is defined to be the direct limit of the equivariant homotopy groups $[S^V \wedge X, S^V \wedge Y]_G$ over all complex representations $V$ of $G$. We denote it by the symbol $\{X, Y\}_G$. As for the non-equivariant case, these are abelian groups. On $\{S^0, S^0\}_G$ there is also a multiplicative structure given by suspension (or equivalently by composition). Again by suspension, every equivariant stable homotopy group $\{X, Y\}_G$ is a module over this ring. G. Segal observed that $\{S^0, S^0\}_G$ is isomorphic to the Burnside ring $A(G)$ of $G$. Hence, every equivariant stable homotopy group naturally is a module over the Burnside ring. More generally, one can show that each equivariant stable homology or cohomology group, or even more generally, each equivariant homology or cohomology group of a theory with Mackey structure, is a module over the respective Burnside ring. In summary, module structures over Burnside rings are ubiquitous in equivariant topology. This corresponds to the general philosophy that the most basic objects in equivariant topology are not single points but orbits; these are just the generators of the Burnside ring.

There is a theorem of Hopf stating in its classical form that the homotopy classes of maps of an $n$-dimensional, connected, closed, oriented manifold into $S^n$ are in one-to-one correspondence with $\mathbb{Z}$ by the mapping degree. It can be generalized to an equivariant situation in order to find conditions for $G$-CW-complexes $X, Y$ such that the equivariant homotopy classes of maps $X \to Y$ are characterized by a generalized mapping degree, [8, II 4.11]. This generalized mapping degree is an element of the ring $C(G)$ of integer valued functions on the set of conjugacy classes of subgroups in $G$. As a consequence, there is an embedding of the equivariant stable homotopy group $\{X, Y\}_G$ into $C(G)$. This leads to the consideration of modules which are finitely generated and free as abelian groups, so called lattices or representations. The terminology coincides with the usual one if we consider a Burnside ring as an order over the ring of integers. This point of view will be explained and used in Section 2.

We concentrate in this article on the investigation of modules of this special type. Unfortunately, general modules do not need to have a decomposition into a representation and a torsion part.

The structure and the main results of this article can be summarized as follows: In the first section we shortly recall the definition and fundamental properties of Burnside rings of finite groups. In Section 2 we prove a necessary and sufficient condition for a Burnside ring to be of finite representation type. As a consequence we see that most groups have a Burnside ring of infinite representation type. For $p$-groups the only exceptions are the cyclic groups of order $p$ and $p^2$. Section 3 contains a more detailed investigation for $p$-groups. In the cases of finite representation type the indecomposable representations are classified. Moreover, we see that for $p$-groups with more than four conjugacy classes of subgroups the representation type of the Burnside ring is wild. The last section collects some nice properties for the special case of groups with square free order.

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## 1. Burnside Rings

For a finite group $G$ the Burnside ring $A(G)$ is defined to be the Grothendieck ring of finite $G$-sets. The sum is given by disjoint union and the product by the direct product of $G$-sets with diagonal action. From this definition it is obvious that Burnside rings are commutative. The unit is represented by a point with trivial action.

Up to isomorphism every $G$-set is a disjoint union of orbits $G/H$. Two such orbits $G/H$ and $G/K$ are isomorphic as $G$-sets if and only if $H$ and $K$ are conjugate in $G$, in symbols $H \sim K$. Therefore, as an abelian group $A(G)$ is free and generated by the isomorphism classes $[G/H]$ with one representative for each conjugacy class of subgroups $H$ in $G$. In particular, Burnside rings are noetherian.

Denote by $\Phi(G)$ the set of conjugacy classes of subgroups in $G$ and by $C(G)$ the ring of integer valued functions on $\Phi(G)$. There is a well defined map

$$A(G) \longrightarrow C(G), \quad S \longmapsto ((H) \mapsto |S^H|).$$

This is an injective ring homomorphism. The image is described by Dress' congruences, [6, Chapter 1.3]. In the following we usually think of $A(G)$ as a subring of $C(G)$.

The ring extension $A(G) \subset C(G)$ is integral. As a consequence, every prime ideal of the Burnside ring is an intersection of $A(G)$ with a prime ideal in $C(G)$, i.e. of the form

$$\mathfrak{q}(H,p) := \{\, z \in C(G) \mid z(H) \equiv 0 \bmod (p) \,\} \cap A(G)$$

for a class $(H) \in \Phi(G)$ and a prime ideal $(p) \subset \mathbb{Z}$. Denote by $O^p(G)$ the unique minimal normal subgroup of $G$ such that the quotient is a $p$-group. We recall a theorem of Dress [9]:

**Proposition 1.1.** *Two prime ideals $\mathfrak{q}(H,p)$ and $\mathfrak{q}(K,r)$ are equal if and only if*

$$p = r \quad \textit{and} \quad \begin{cases} (H) = (K) & \textit{if } \ p = r = 0, \\ O^p(H) \sim O^p(K) & \textit{if } \ p = r \textit{ is prime.} \end{cases}$$