ON GOOD MODULE CATEGORIES WITHOUT SHORT CYCLES

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Abstract. Let A be a quasi–hereditary algebra, and $\mathcal{F}(\Delta)$ the good module category consisting of A–modules which have a filtration by standard modules. An indecomposable module M in $\mathcal{F}(\Delta)$ is said to be on a short cycle in $\mathcal{F}(\Delta)$ if there exists an indecomposable module N in $\mathcal{F}(\Delta)$ and a chain of two nonzero noninvertible maps $M \to N \to M$. It is shown that two indecomposable modules in $\mathcal{F}(\Delta)$ are isomorphic if they are not on short cycles in $\mathcal{F}(\Delta)$ and have the same composition factors. Moreover, if there is no short cycle in $\mathcal{F}(\Delta)$, we show that $\mathcal{F}(\Delta)$ is finite, that is, there are only finitely many isomorphism classes of indecomposables in $\mathcal{F}(\Delta)$. This is an analogue to a result in a complete module category proved by Happel and Liu.

Introduction and Preliminaries

Let A be an artin algebra over a commutative artin ring R, $\operatorname{mod} A$ the category of finitely generated left A–modules. A cycle in $\operatorname{mod} A$ is a sequence $X_0 \to X_1 \to \cdots \to X_n = X_0$ of nonzero noninvertible maps between indecomposable A–modules. Such a cycle is said to be short if $n = 2$. Cycles in $\operatorname{mod} A$ have been widely studied (see, for example, [8], [1], [11], [6], [5]). In particular, in [5] it is proved that A is of finite type if $\operatorname{mod} A$ contains no short cycles.

In the study of a quasi–hereditary algebra A, instead of the complete module category $\operatorname{mod} A$, one is mainly interested in the category $\mathcal{F}(\Delta)$ consisting of A–modules which have a filtration by standard modules. It is proved by Ringel in [9] that $\mathcal{F}(\Delta)$ has almost split sequences. Thus the usual techniques of representation theory can be adapted. In this paper we consider the notion of a short cycle in $\mathcal{F}(\Delta)$, i.e. a chain of nonzero noninvertible maps $M \to N \to M$ with M and N indecomposable in $\mathcal{F}(\Delta)$. We will show that an analogous result to the one of

Happel and Liu in [5] holds for $\mathcal{F}(\Delta)$, namely, if $\mathcal{F}(\Delta)$ contains no short cycles, then $\mathcal{F}(\Delta)$ is finite.

We recall now the definition of a quasi–hereditary algebra and some basic facts (see [3] and [9]). Let A be an artin algebra. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the full subcategory of all A–modules which have a Θ–filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to one object in Θ for $1 \leq i \leq t$. The modules in $\mathcal{F}(\Theta)$ are called Θ-good modules, and the category $\mathcal{F}(\Theta)$ is called the Θ-good module category.

Let $E(\lambda)$, $\lambda \in \Lambda$, be a complete set of simple A–modules, where Λ is a finite partial ordered set. For each $\lambda \in \Lambda$, let $P(\lambda)$ be (or $P_A(\lambda)$) a projective cover of $E(\lambda)$ and denote by $\Delta(\lambda)$ the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Dually, let $Q(\lambda)$ (or $Q_A(\lambda)$) be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with the composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Let Δ (respectively, ∇) be the full subcategory consisting of all $\Delta(\lambda)$, $\lambda \in \Lambda$ (respectively, all $\nabla(\lambda)$, $\lambda \in \Lambda$). We call modules in Δ standard modules and ones in ∇ costandard modules.

The algebra A, or better the pair (A, Λ) is called quasi–hereditary if $\mathrm{End}_A(\Delta(\lambda))$ is a division ring, for each $\lambda \in \Lambda$ and the module ${}_AA$ belongs to $\mathcal{F}(\Delta)$.

For a quasi–hereditary algebra A with respect to a poset Λ we call the elements in Λ weights and Λ the weight poset of A. The category $\mathcal{F}(\Delta)$ is called the good module category of the algebra A and admits the following description

$$\begin{aligned}\mathcal{F}(\Delta) &= \{X \in \mathrm{mod}\, A | \mathrm{Ext}^1_A(X, \nabla) = 0\} \\ &= \{X \in \mathrm{mod}\, A | \mathrm{Ext}^i_A(X, \nabla) = 0 \ \text{ for all } i \geq 1\}.\end{aligned}$$

Dually, one has that

$$\begin{aligned}\mathcal{F}(\nabla) &= \{Y \in \mathrm{mod}\, A | \mathrm{Ext}^1_A(\Delta, Y) = 0\} \\ &= \{X \in \mathrm{mod}\, A | \mathrm{Ext}^i_A(\Delta, Y) = 0 \ \text{ for all } i \geq 1\}.\end{aligned}$$

Thus $\mathcal{F}(\Delta)$ is closed under kernels of surjective maps, and $\mathcal{F}(\nabla)$ is closed under cokernels of injective maps.