Characteristic tilting modules over a class of quasi–hereditary algebras

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Abstract

With each finite directed quiver Q a quasi–hereditary algebra, the so–called twisted double of the path algebra kQ, is associated (This class of algebras contains certain interesting ones). Charateristic tilting modules over this class of quasi–hereditary algebras are constructed. Their endomorphism algebras are explicitly described. It turns out that this class of quasi–hereditary algebras is closed under taking Ringel dual.

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Quasi–hereditary algebras are introduced by Scott in [15] in order to study highest weight categories in representation theory of semisimple Lie algebras and algebraic groups (see also [3]). Many important algebras such as hereditary algebras, Schur algebras, and algebras to blocks of the category $\mathcal{O}$ in [2] are typical examples of quasi–hereditary algebras.

In the study of a quasi–hereditary algebra A (with respect to a partial order on the set Λ of simple A–modules), the central notions are the standard modules $\{\Delta(\lambda) : \lambda \in \Lambda\}$ and the costandard modules $\{\nabla(\lambda) : \lambda \in \Lambda\}$. Of main interest are the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ of A–modules which have a Δ-filtration and ∇-filtration respectively. It is proved by Ringel in [14] that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ contains only finitely many isomorphism classes of indecomposable modules, which may be labelled by $T(\lambda)$, $\lambda \in \Lambda$. These modules $T(\lambda)$, which now are called tilting modules, soon become an important object in representation theory of Lie algebras, algebraic groups and quantum groups (see, for example, [7], [8], [1] and [13]). Moreover, the module $T := \oplus T(\lambda)$ is called the characteristic tilting module of A whose endomorphism algebra, called the Ringel dual of A, is again quasi–hereditary.

In general, it seems to be difficult to describe characteristic tilting modules and their endomorphism algebras. However, these questions could be dealt with for some special classes of quasi–hereditary algebras (see, for example, [4], [5], [7], [8], [12] and

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[17]). In the present paper we deal with a class of quasi–hereditary algebras which are twisted doubles of hereditary algebras (This class of algebras covers quadratic duals of all Temperley–Lieb algebras). The characteristic tilting modules and the Ringel duals of these algebras are explicitly described. It turns out that this class of quasi–hereditary algebras is closed under taking Ringel dual. This generalizes the main results in [4, Section 4]. Moreover, the method presented in this paper simplifies the proofs there.

1. Preliminaries

In this section we recall some definitons and include some basic results needed in the sequel.

1.1 Let A be a finite dimensional algebra over a field k. The category of all finitely generated left A–modules will be denoted by A–mod, and the composition of maps $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ is denoted by gf. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the full subcategory of all A–modules which have a Θ–filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to one object in Θ for $1 \leq i \leq t$. The modules in $\mathcal{F}(\Theta)$ are called Θ-good modules, and the category $\mathcal{F}(\Theta)$ is called the Θ-good module category.

Let Λ be a finite poset in bijective correspondence with the isomorphism classes of simple A–modules. For each $\lambda \in \Lambda$, let $E(\lambda)$ be a simple module in the isomorphism class corresponding to λ and $P(\lambda)$ (or $P_A(\lambda)$) a projective cover of $E(\lambda)$ and denote by $\Delta(\lambda)$ the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Dually, let $Q(\lambda)$ (or $Q_A(\lambda)$) be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with the composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Let Δ (respectively, ∇) be the full subcategory consisting of all $\Delta(\lambda)$, $\lambda \in \Lambda$ (respectively, all $\nabla(\lambda)$, $\lambda \in \Lambda$). We call modules in Δ standard modules and ones in ∇ costandard modules.

The algebra A is said to be quasi–hereditary with respect to $(\Lambda, \leq)$ if for each $\lambda \in \Lambda$ we have

(i) $\mathrm{End}_A(\Delta(\lambda))$ is a division ring;

(ii) $P(\lambda) \in \mathcal{F}(\Delta)$, and moreover, $P(\lambda)$ has a Δ–filtration with factors $\Delta(\mu)$ for $\mu \geq \lambda$ in which $\Delta(\lambda)$ occurs exactly once.

For a quasi–hereditary algebra A with respect to a poset Λ we call the elements in Λ weights and Λ the weight poset of A. By (A, Λ) we denote a quasi–hereditary algebra A with the weight poset Λ.

If a quasi–hereditary algebra has a duality δ on the category A–mod which fixes simple modules, we call it a BGG–algebra (see [9], [3]).

1.2 Let A be a quasi–hereditary algrbra with the weight poset Λ. By [14], there are indecomposable modules $T(\lambda)$, $\lambda \in \Lambda$, and exact sequences

$$0 \longrightarrow \Delta(\lambda) \longrightarrow T(\lambda) \longrightarrow X(\lambda) \longrightarrow 0$$