The Δ-filtered modules without self-extensions for the Auslander algebra of $k[T]/\langle T^n \rangle$

Thomas Brüstle, Lutz Hille, Claus Michael Ringel, Gerhard Röhrle

ABSTRACT. It is well-known that the Auslander algebra of any representation finite algebra is quasi-hereditary. We consider the Auslander algebra A_n of $k[T]/\langle T^n \rangle$ (here, k is a field, T a variable and n a natural number). We determine all Δ-filtered A_n-modules without self-extensions; they can be described purely combinatorially. We show, given any Δ-filtered module N, there is (up to isomorphism) a unique Δ-filtered module M without self-extensions which has the same dimension vector. In case k is an infinite field, N is a degeneration of this module M. In particular, we see that in this case, the set of Δ-filtered modules with a fixed dimension vector is the closure of an open orbit (thus irreducible). As observed in [HiRö], the problem of describing all Δ-filtered A_n-modules is the same as that of describing the conjugacy classes of elements in the unipotent radical of a parabolic subgroup P of $\mathrm{GL}(m,k)$ under the action of P, thus we recover Richardson's dense orbit theorem in this instance.

1. Preliminaries.

Let k be a field. We consider k-algebras, their modules, and maps between these modules. All modules are left modules, and all maps are written on the left of the argument. Thus, if X, Y and Z are modules, the composition of homomorphisms $f: X \to Y$ and $g: Y \to Z$ is denoted by gf.

Let $R_n = k[T]/\langle T^n \rangle$, this is a representation finite algebra with precisely n isomorphism classes of indecomposable representations $M(i)$, $1 \le i \le n$, and it is convenient to choose the indexing[1] so that $M(i)$ is of length $n - i + 1$.

Let $A_n = \mathrm{End}(\bigoplus_i M(i))$ be the Auslander-algebra of R_n with the corresponding indexing of the simple A_n-modules, thus the indecomposable projective A_n-modules $P(i)$ embed into each other as follows

$$P(1) \supset P(2) \supset \cdots \supset P(n-1) \supset P(n).$$

Note that A_n is quasi-hereditary in a unique way, its weight set is the canonically ordered set $\{1, 2, \ldots, n\}$. We denote by $E(i)$ the simple factor module of $P(i)$ and we stress that the module $P(1)$ is just the injective envelope of $E(1)$.

We recall some well-known properties (see Lemma 7.1 of [DlRi]): For $1 \le i \le n$, the standard A_n-module $\Delta(i)$ is the serial module of length i with socle $E(1)$. The class $\mathcal{F}(\Delta)$ of all Δ-filtered (or Δ-good) modules is just the set of all modules with socle generated by $E(1)$. In particular, $\mathcal{F}(\Delta)$ is closed under submodules and all projective A_n-modules belong to $\mathcal{F}(\Delta)$.

For any Δ-filtered module M, we denote by $(\mathbf{dim}_\Delta\, M)_i$ the multiplicity of $\Delta(i)$ in a Δ-filtration of M (these multiplicities are uniquely determined, since we deal

[1] With respect to the modules themselves, it would be more natural to use the actual length as index. However, this would lead to a conflict with respect to the traditional way of using indices for the weight set of a quasi-hereditary algebra: we would have to endow a sequence of natural numbers with the ordering opposite to the natural one.

with the standard modules of a quasi-hereditary algebra). The (ordered) n-tuple $\mathbf{dim}_\Delta M$ with coefficients $(\mathbf{dim}_\Delta M)_i$ is said to be the Δ-*dimension vector* of M. The Δ-*support* of M is the set of indices i such that $(\mathbf{dim}_\Delta M)_i \neq 0$.

The dual statements are also of interest: For $1 \leq i \leq n$, the costandard A_n-module $\nabla(i)$ is the serial module of length i with top $E(1)$. The class $\mathcal{F}(\nabla)$ of all ∇-filtered modules is just the set of all modules with top generated by $E(1)$. Similar to the Δ-dimension vector and the Δ-support of modules in $\mathcal{F}(\Delta)$, there is a notion of ∇-dimension vector and ∇-support for any module in $\mathcal{F}(\nabla)$, using filtrations by costandard modules.

We denote by $T(i)$ the unique indecomposable module[2] which has both a Δ-filtration and a ∇-filtration, one composition factor of the form $E(i)$, and all other composition factors of the form $E(j)$ with $j < i$. Note that $T(n) = P(1)$, this module is both projective and injective.

Lemma 1. *The modules in $\mathcal{F}(\Delta)$ are just the A_n-modules with projective dimension at most 1.*

Proof: In order to bound the projective dimension of the modules in $\mathcal{F}(\Delta)$, it is sufficient to consider the standard modules. But $\Delta(n) = P(n)$, and for $1 \leq i < n$, a projective resolution of $\Delta(i)$ is given by

$$0 \to P(i+1) \to P(i) \to \Delta(i) \to 0.$$

This shows that the projective dimension of any module in $\mathcal{F}(\Delta)$ is at most 1.

Conversely, assume that M is an A_n-module with a projective resolution $0 \to P_1 \to P_0 \to M \to 0$. We want to show that the socle of M is the direct sum of copies of $E(1)$. Let S be a simple submodule of M. The inclusion map $S \subseteq M$ yields an induced exact sequence $0 \to P_1 \to X \to S \to 0$, where X is a submodule of P_0. If this induced sequence splits, then S is isomorphic to a submodule of X, thus of P_0, and therefore isomorphic to $E(1)$. If it does not split, then $\mathrm{Ext}^1(S, P_1) \neq 0$, thus there is an indecomposable projective module $P(i)$ with $\mathrm{Ext}^1(S, P(i)) \neq 0$ (and $i > 1$). Note that $P(i)$ embeds into $P(1)$ and that the quotient is just $T(i-1)$. If we apply $\mathrm{Hom}(S, -)$ to the exact sequence $0 \to P(i) \to P(1) \to T(i-1) \to 0$, we obtain an exact sequence

$$\mathrm{Hom}(S, T(i-1)) \to \mathrm{Ext}^1(S, P(i)) \to 0,$$

where we use again that $P(1)$ is injective. But the only simple module S with $\mathrm{Hom}(S, T(i-1)) \neq 0$ is $S = E(1)$. This completes the proof.

2. The modules $\Delta(I)$ and $\Delta(\mathbf{d})$.

In this section we define some Δ-filtered modules which fail to admit any self-extension. The indecomposable ones are labelled $\Delta(I)$, where I is a non-empty subset of $\{1, \ldots, n\}$. We also consider suitable direct sums of such modules $\Delta(I)$:

[2] Occasionally, these modules are referred to as "tilting modules". However, this is in conflict with the original terminology. The tilting modules we are going to consider in Section 7 are the traditional ones.