ON THE AUSLANDER-REITEN QUIVER OF AN INFINITESIMAL GROUP

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0. Introduction and Preliminaries

The investigation of the representations of finite group schemes over algebraically closed fields of positive characteristic typically proceeds in several steps. By general theory (cf. [29, (6.8)]) a finite algebraic group scheme $\mathcal{G}$ decomposes into a semidirect product $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{red}$ with a normal infinitesimal subgroup $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{red}$. Accordingly, one begins by studying the representation theories of the constituents. These turn out to differ with regard to both, their methods and results. The final, and often most difficult, step amounts to fusing the results for infinitesimal and constant groups.

The classical modular representation theory of finite groups is fairly completely developed. One understands the Morita equivalence classes of the representation-finite and tame blocks (cf. [3, 7]), and the Auslander-Reiten quiver has also been determined (cf. [8]). Much of the success here rests on the availability of a comprehensive block theory that is based on Green's theory of vertices and sources.

By contrast, much less is known for the group algebras of infinitesimal algebraic groups. A general block theory remains elusive, and the most promising approach pursued so far is based on geometric methods related to the notion of a rank variety. In recent work Suslin-Friedlander-Bendel [26, 27] have extended this notion from infinitesimal groups of height ≤ 1 to those of arbitrary height. The main purpose of the present note is to show how their results and those of [12, 17] can be exploited to study the stable Auslander- Reiten quiver of an infinitesimal group.

The paper is organized as follows. In the first section we show that the rank variety of an indecomposable module is an invariant of its stable Auslander-Reiten component. By combining work of Happel-Preiser-Ringel [20] with that of Erdmann-Skowroński [10] one then readily obtains the standard list of the possible tree classes of the AR-components: finite

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and infinite Dynkin diagrams, and Euclidean diagrams. Thus, Theorem 1.3 is the analogue of Webb's theorem [30] for group algebras of finite groups (see [9, 11] for infinitesimal groups of height ≤ 1). In §2 we investigate the question to what extent the structure of an AR-component is determined by the dimension of its support variety. Components with one-dimensional variety are those containing periodic modules and are therefore either finite, or infinite tubes. In generalization of the main result of [13] we show that AR-components with support varieties of dimension ≥ 3 are isomorphic to $\mathbb{Z}[A_\infty]$.

Following a brief discussion on the number of AR-components, we turn in the last two sections to special cases given by Frobenius kernels of reductive groups and supersolvable groups. For the former actions on varieties provide additional information, while a certain "linkage principle" is exploited in the investigation of the latter. For both classes of groups, one can reduce the list of possible tree classes by determining the finite and Euclidean components.

Throughout this paper we will be working over an algebraically closed field k of characteristic $p > 0$. All k-vector spaces are assumed to be of finite dimension. Let H be a Hopf algebra with counit $\varepsilon : H \longrightarrow k$ and antipode $\eta : H \longrightarrow H$. According to a result due to Sweedler (cf. [28, (5.1.6)]), there exists a one-dimensional subspace $\int_H^r \subset H$, the space of *right integrals* of H, such that

$$x \cdot h = x\,\varepsilon(h) \quad \forall\, h \in H,\ x \in \int_H^r .$$

The unique algebra homomorphism $\zeta : H \longrightarrow k$ satisfying $h \cdot x = \zeta(h)\,x$ for $h \in H$ and $x \in \int_H^r$ is called the *modular function* of H.

Given an algebra homomorphism $\lambda : H \longrightarrow k$ the convolution

$$\lambda * id_H : H \longrightarrow H \quad ; \quad h \mapsto \sum_{(h)} \lambda(h_{(1)})h_{(2)}$$

is an automorphism of H of finite order. Every Hopf algebra is a (not necessarily symmetric) Frobenius algebra (cf. [24]). In case H is cocommutative, $\nu := \zeta * id_H$ is a Nakayama automorphism of H (cf. [18, (1.5)]).

For an H-module M and an automorphism $\gamma : H \longrightarrow H$ of k-algebras we denote by $M^{(\gamma)}$ the H-module with underlying k-space M and action given by

$$h \cdot m := \gamma^{-1}(h)m \quad \forall\, h \in H,\, m \in M.$$

In this fashion we obtain an action of the automorphism group $Aut_k(H)$ on mod_H, the category of H-modules. If $\lambda : H \longrightarrow k$ is an algebra homomorphism, then k_λ denotes the corresponding one-dimensional H-module. Note that $\lambda \circ \eta$ is also a homomorphism, and that $M^{(\lambda * id_H)} \cong M \otimes_k k_{\lambda\circ\eta}$. In particular, we have $M^{(\nu)} \cong M \otimes_k k_{\zeta\circ\eta}$.

1. The Variety of an Auslander-Reiten Component

We refer the reader to [6, 22, 29] for general facts on algebraic k-groups. Given such a group $\mathcal{G}$, the r^{th} Frobenius kernel of $\mathcal{G}$ will be denoted $\mathcal{G}_r$. If $\mathcal{G} = Spec_k(\mathcal{O}(\mathcal{G}))$ is a finite algebraic