A generalization of Hasminskii's theorem on existence of invariant measures for locally integrable drifts

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Abstract

We prove existence of invariant measures for diffusions on $\mathbb{R}^n$ whose drift are merely locally in L^p with $p > n$. Our only global assumption is that a Lyapunov-type function exists. Subsequently, we generalize this result to the case where $\mathbb{R}^n$ is replaced by a Riemannian manifold.

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0 Introduction

A well-known theorem due to Hasminskii [1], [2] states that a diffusion process in $\mathbb{R}^n$ (or, more generally, in a smooth manifold) with locally Lipschitzian coefficients possesses an invariant probability provided there exists a certain Lyapunov-type function for the generator of this diffusion. In applications one of the best known cases is where the drift term is coercive. In recent years there has been a growing interest in diffusion processes with more general, possibly singular drifts. It has been discovered that this more general situation preserves many essential features of the locally Lipschitzian case. A generalization of Hasminskii's theorem has been obtained in [3, §4.3] for nondegenerate diffusion processes with continuous coefficients and certain classes of Lyapunov functions. Further extensions are found in [4], [5], [6], where the existence of invariant probabilities is established for diffusion

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processes with nondegenerate continuous diffusion terms satisfying certain uniform estimates and with locally bounded coercive drifts. On the other hand, it has been shown in [7] that any invariant measure of a nondegenerate diffusion in $\mathbb{R}^n$ (or in a smooth n-dimensional manifold) with drift locally integrable of order more than n admits a density locally in a Sobolev space from the same class as in the locally Lipschitzian case; under the same condition Varadhan's conjecture concerning the uniqueness of invariant probabilities for diffusions with non-smooth drifts has been settled positively in [8]. The purpose of this note is to apply the methods used in [7], [8] to replace the local Lipschitzness (or local boundedness) of the drift in the aforementioned theorem of Hasminskii by its local integrability. The formulations and proofs of our results are essentially analytical which seems to be natural from the point of view of the problems we consider; however, their probabilistic interpretation is straightforward. It is well-known that in the case of locally Lipschitzian coefficients any invariant measure μ for the diffusion ξ governed by the stochastic differential equation

$$d\xi_t = \sqrt{2A(\xi_t)}\, dw_t + B(\xi_t)dt$$

satisfies the elliptic equation

$$L^*_{A,B}\mu = 0 \tag{0.1}$$

with

$$L_{A,B}f = \mathrm{trace}(Af'') + (B, \nabla f), \quad f \in C_0^\infty(\mathbb{R}^n), \tag{0.2}$$

in the following weak sense:

$$\int_{\mathbb{R}^n} L_{A,B}f\, d\mu = 0, \quad \forall\, f \in C_0^\infty(\mathbb{R}^n), \tag{0.3}$$

where implicitly we always assume that $|B|$, $A^{ij} \in L^1_{loc}(\mu)$ with $A = (A^{ij})$.

In many applications it becomes more convenient to work with the weak elliptic equation (0.3) instead of the invariance of measures in the sense of semigroups. However, as it will be mentioned below, under broad conditions (including those used in our paper) these two settings are equivalent.

For every $R > 0$ set $U_R := \{x \in \mathbb{R}^n : |x| < R\}$. If Ω is an open set in $\mathbb{R}^n$, we denote by $H^{p,1}(\Omega)$ the Sobolev class of functions which together with their generalized first order derivatives belong to $L^p(\Omega)$. The class $H^{p,1}(\Omega)$ is equipped with the Sobolev norm

$$\|f\|_{H^{p,1}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{i=1}^n \|\partial_{x_i} f\|_{L^p(\Omega)}.$$

The symbol $C_0^\infty(\Omega)$ stands for the class of all infinitely differentiable functions on Ω with compact supports in Ω. Analogously, one defines the class $C_0^\infty(M)$ for a smooth finite dimensional manifold M. By $H^{p,1}_{loc}(\Omega)$ we denote the class of all functions f on Ω such that $\zeta f \in H^{p,1}(\Omega)$ for every $\zeta \in C_0^\infty(\Omega)$. Norms of vectors are denoted by $|\cdot|$ and norms of operators are denoted by $\|\cdot\|$.