Chromatic-Index Critical Multigraphs of Order 20

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Abstract

A multigraph M with maximum degree $\Delta(M)$ is called critical, if the chromatic index $\chi'(M) > \Delta(M)$ and $\chi'(M-e) = \chi'(M) - 1$ for each edge e of M. The weak critical graph conjecture [1, 7] claims that there exists a constant $c > 0$ such that every critical multigraph M with at most $c \cdot \Delta(M)$ vertices has odd order. We disprove this conjecture by constructing critical multigraphs of order 20 with maximum degree k for all $k \geq 5$.

1 Introduction

We consider connected multigraphs without loops. Throughout the paper we denote the edge set and the vertex set of a multigraph M by $E(M)$ and $V(M)$, the degree of a vertex v of M by $d_M(v)$, and the maximum degree of M by $\Delta(M)$. If $m \geq 1$ edges join the same pair v, w of vertices of M, then vw is called a *multiple edge* of *multiplicity* m or an m*-edge*. A *proper multiple edge* is a multiple edge of multiplicity at least 2 and a *graph* is a multigraph without proper multiple edges. An r*-colouring* of M is a function $\pi : E(M) \to \{1, ..., r\}$ so that $\pi(e) \neq \pi(f)$ for adjacent edges e and f. The *chromatic index* $\chi'(M)$ of a multigraph M is the smallest number r so that there is an r-colouring of M. A multigraph M is called critical, if $\chi'(M) > \Delta(M)$ and $\chi'(M-e) = \chi'(M) - 1$ for each edge e of M. We define M to be k*-critical*, if it is critical and $k = \Delta(M) = \chi'(M) - 1$. The Petersen graph will be denoted by P, and the graph obtained from P by deleting a vertex is called $P - v$.

Jakobsen [6] formulated the famous critical graph conjecture, which claims that every critical multigraph M has odd order. This conjecture was disproved by Goldberg [3] for $\Delta(M) = 3$, and Chetwynd and Fiol independently disproved it for $\Delta(M) = 4$ (see [1]). Recently, the author and Steffen [4] constructed k-critical graphs and multigraphs of even order for all $k \geq 4$. After the critical graph conjecture had been disproved, a new conjecture appeared in a paper by Chetwynd and Wilson [1].

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Conjecture 1.1 (Weak Critical Graph Conjecture) *There exists a constant $c > 0$ such that every critical multigraph M with at most $c \cdot \Delta(M)$ vertices has odd order.*

According to Chetwynd and Wilson [1], Yap [8], and Jensen and Toft [7], this conjecture was formulated by Goldberg [2], but in fact it does not appear in Goldberg's paper.

We disprove Conjecture 1.1 by proving the following theorem.

Theorem 1.2 *For all $k \geq 5$, there exists a k-critical multigraph of order 20.*

2 Basic Theorems

In this section we present several tools for constructing k-critical multigraphs.

Let M be a multigraph and S a set of edges of M. We say that N is obtained from M by *adding* S if the multiplicity of each edge of S increases by 1.

Lemma 2.1 ([4]) *Let $k \geq 0$ and M be a multigraph obtained from P by adding k 1-factors of P. Then $M - v$ is a $(k+3)$-critical multigraph for any $v \in V(M)$.*

A multigraph N is said to be obtained from M and M' by *Hajós-union* where v and v' are identified, if $v \in V(M)$, $v' \in V(M')$, and N is constructed from M and M' as follows:

1. One edge $uv \in E(M)$ and one edge $u'v' \in E(M')$ are removed.
2. The vertices v and v' are identified.
3. A new edge uu' is inserted.

Lemma 2.2 ([5]) *Let M and M' be two k-critical multigraphs and $v \in V(M)$, $v' \in V(M')$ two vertices such that $d_M(v) + d_{M'}(v') \leq k+2$. Then any multigraph obtained from M and M' by Hajós-union where v and v' are identified is k-critical.*

Let M be a multigraph on vertices $v_1, ..., v_n$ and with maximum vertex degree k. We call $s(M) = kn - \sum_{i=1}^{n} d_M(v_i)$ the *deficiency* of M.

Lemma 2.3 ([4]) *Let M be a k-critical multigraph, $k \geq 2$, and N an induced proper submultigraph of odd order. If $s(N) = k$ then the multigraph M^- obtained from M by contracting N to a single vertex is k-critical.*

Let M be a multigraph. We say that M^+ is obtained from M by *planting* an m-edge on the edge $uv \in E(M)$ if M^+ is obtained from M by removing uv and inserting vertices u^+, v^+ and the edges uu^+, vv^+ and an m-edge u^+v^+.

Lemma 2.4 *Let M be a k-critical multigraph, $k \geq 2$, and let M^+ be a multigraph obtained from M by planting a $(k-1)$-edge on an edge $uv \in E(M)$. Then M^+ is a k-critical multigraph.*