THE MILGRAM NON-OPERAD

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Introduction

In [11] R.J. Milgram introduced geometric models J^n for the iterated loop-space operads $\Omega^n\Sigma^n$. Later J.M. Boardman, R.M. Vogt and P. May proved that n-fold loop spaces are closely related to E_n operads in general, and thus to the little cube operads (see [6], [7], [9]). Hence the question arises if the operad structure of $\Omega^n\Sigma^n$ translates to the geometric model J^n.

In [1] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R.M. Vogt construct an operad $\mathcal{M}_n$, which codifies n-fold monoidal categories, a categorial analog of n-fold loop spaces. They observe that an equivalent *preoperad* is embedded in $\mathcal{M}_n$, whose free space is of the same homotopy type as J^nX . For $n = 2$ the spaces are even homeomorphic (see [1, 3.12 – 14]).

Due to the underlying polytopes, the permutohedra, the Milgram-construction J^n is of some importance for the examination of coherent homotopy-commutativity. Similar to the associahedra introduced by Stasheff in [13], Williams uses the permutohedra and the Milgram-construction in [14] to define his notion of C_n-spaces, which is used in several subsequent papers, and is occuring in papers of McGibbon and Hemmi (see for example [10] and [8]).

In fact there exists an operad structure with the permutohedra as underlying spaces (this was pointed out to me by Clemens Berger and Zig Fiedorowicz), simply by using the convex extension of the permutation operad. But since the permutohedra are contractible this is an E_∞ operad in the sense of Boardman and Vogt (i.e. the symmetric group action does not need to be free). Therefore an algebra of this operad is homotopy equivalent to an infinite loop space and hence its associated monad can not be of the same homotopy type as the Milgram construction J^n which is just an n-fold loop space.

In [4] and [5] C. Berger conjectured an E_n operad structure of the permutohedra, whose associated monad is the Milgram construction J^n.

I will show that Berger's construction does not work. The first observation is that the would-be operad bears a structure far too strong, namely that of strictly abelian monoids. This collapse of structure is then used to show that the suggested structure does not define an operad at all. In fact the proof shows that the multiplication defined by Berger does not respect the degeneration conditions.

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1. The Permutohedra and Berger's construction

Let Λ be the category of finite sets $\mathbf{n} = \{1, \ldots, n\}$ and injective maps. Each injective map $\varphi : \mathbf{n} \to \mathbf{m}$ has an unique decomposition of the form $\varphi = \varphi^{inc} \circ \varphi^\natural$, such that $\varphi^{inc} : \mathbf{n} \to \mathbf{m}$ is increasing and $\varphi^\natural : \mathbf{n} \to \mathbf{n}$ is a permutation.

Definition 1.1. The *Permutation preoperad* $\Sigma : \Lambda^{op} \to \mathbf{Top}$ is the functor with $\Sigma(\mathbf{n}) = \Sigma_n$ and $\varphi^* := \Sigma(\varphi) : \Sigma_m \to \Sigma_n$ given by $\sigma \mapsto (\sigma \circ \varphi)^\natural$ for $\varphi \in \Lambda(\mathbf{n}, \mathbf{m})$

The multiplications of the *Permutation Operad* Σ are given by

$$\begin{aligned} \gamma^\Sigma_{n;i_1,\dots,i_n} : \Sigma_n \times \Sigma_{i_1} \times \cdots \times \Sigma_{i_n} &\to \Sigma_{i_1+\cdots+i_n} \\ (\sigma; \tau_1, \dots, \tau_n) &\mapsto \sigma(i_1, \dots, i_n) \circ (\tau_1 \oplus \cdots \oplus \tau_n) \end{aligned}$$

where $\sigma(i_1, \dots, i_n)$ permutes the blocks (see [5, 1.15.(a)]) and $- \oplus - : \Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ is the canonical product of permutations.

The product $\Sigma_{i_1} \oplus \cdots \oplus \Sigma_{i_n} \subset \Sigma_{i_1+\cdots+1_n}$ will be denoted with $\Sigma(i_1, \dots, i_n)$.

Like Berger I will use the right action of the symmetric groups on spaces. One can switch to the left action by using the inverse permutations instead. For more details about (pre)operads in general the reader is referred to [4], [5] or [9].

Definition 1.2. For $n \geq 1$ the symmetric group Σ_n acts on $\mathbb{R}^n$ from the right by permuting the coordinates

$$(x_1, \dots, x_n)\sigma = \big(x_{\sigma(1)}, \dots x_{\sigma(n)}\big).$$

The *n-th Permutohedron* $P_n \subset \mathbb{R}^n$ is the convex hull of the orbit of $(1, 2, \dots, n) \in \mathbb{R}^n$ under this operation.

The convex hull of the orbit of $\Sigma(i_1, \dots, i_r) \subset \Sigma_n$ will be denoted with $P(i_1, \dots, i_r) \subset P_n$.

The geometric and simplicial properties of these polygons were examined in [11], [14] and [2]. Here I will give only a rough sketch of the few details I will use.

P_1 consists of only one point, P_2 is homeomorphic to the unit interval in $\mathbb{R}$ and P_2 to the hexagon in $\mathbb{R}^2$. In general P_n is a $(n-1)$ dimensional polytope.

Remark 1.3. In [11] and [12] Milgram called the n-th permutohedron $C(n-1)$ instead of P_n.

Obviously there exists a right Σ_n-action on P_n given by permuting the coordinates. The vertices are mapped to vertices, and for each $\sigma \in \Sigma_n$ the map $P_n \to P_n$ with $x \mapsto x\sigma$ is a homeomorphism. But unfortunaltely this action is not free, since the barycenter of each permutohedron is a fixed point.

An arbitrary point $x \in P_n$ will be denoted by a linear combination of permutations

$$x = \sum_{\sigma\in\Sigma_n} t_\sigma \sigma \qquad \text{with} \qquad \sum_{\sigma\in\Sigma_n} t_\sigma = 1.$$

If $x = \sum_{\sigma\in\Sigma_n} s_\sigma \sigma$ and $y = \sum_{\tau\in\Sigma_m} t_\tau \tau$ are points of P_n resp. P_m, the point $x \oplus y \in P_{n+m}$ is given by

$$x \oplus y = \sum_{\sigma\in\Sigma_n} \sum_{\tau\in\Sigma_m} s_\sigma t_\tau \sigma \oplus \tau.$$

In [11] Milgram defined maps $I_k : P_k \times P_{n-k} \to P_n$ given by $(x, y) \mapsto x \oplus y$ mapping the product of two permutohedra into certain faces of a higher dimensional permutohedron. More general the codimension $(r-1)$ faces of P_n are in one-to-one correspondence with the ordered partitions of $\{1, \dots, n\}$ of type $(i_1, \dots, i_r)$ with $i_1 + \cdots + i_r = n, i_k \geq 1$.

Remark 1.4. Each partition of type $(i_1, \dots, i_r)$ can be interpreted as a permutation of $i_1 + \cdots + i_r$ elements. If the classes are given by

$$\{j_1, \dots, j_{i_1}\}, \{j_{i_1+1}, \dots, j_{i_1+i_2}\}, \dots, \{j_{i_1+\cdots+i_{r-1}+1}, \dots, i_{j_1+\cdots+j_r}\}$$