The conformal bratwurst map and polynomial iterative methods

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Abstract

Conformal maps from the exterior of the closed unit disk onto the exterior of 'bratwurst' shape sets in the complex plane are constructed. Using these maps, coefficients for the computation of the corresponding Faber polynomials are derived. The zeros of the Faber polynomials are studied numerically. A 'bratwurst' shape set is the result of deforming an ellipse with foci on the real axis, by conformally mapping the real axis onto the unit circle. Such sets are well suited to serve as general purpose sets for inclusion of sets associated with a matrix, for example the spectrum, field of values or a pseudospectrum. Hence, the sets can be applied in the construction and analysis of polynomial iterative methods for the solution of linear systems. The main advantage of the approach is that the conformal maps are derived from elementary transformations, allowing an easy computation of the transfinite diameter, asymptotic convergence factor and Faber polynomials associated with the 'bratwurst' shape sets. Numerical examples are given.

Key words. polynomial iterative methods, linear systems, convergence analysis, hybrid methods, Faber polynomials, conformal mapping, transfinite diameter

AMS Subject Classifications. 65F10, 30C20, 30C30, 30E10

1 Introduction

In recent years, classical methods of approximation theory have been of considerable interest in the theory of polynomial iterative methods (PIMs) for nonsingular systems of linear equations,

$$A\hat{x} = b,\ A \in \mathbf{C}^{N \times N},\ b \in \mathbf{C}^{N}. \tag{1.1}$$

Given an initial guess x_0 for the solution of (1.1), a PIM is an iterative method yielding iterates x_n which satisfy

$$x_n = x_0 + q_n(A) r_0, \tag{1.2}$$

where $r_0 := b - Ax_0$ is the initial residual and the iteration polynomial $q_n(z)$ is of degree at most $n-1$. The nth residual satisfies

$$r_n := b - Ax_n = p_n(A) r_0,\ \ p_n(z) := 1 - z q_n(z) \in \pi_n, \tag{1.3}$$

where π_n denotes the set of polynomials of degree at most n normalized to have value one at the origin. $p_n(z)$ is called the nth residual polynomial of the PIM, and should be chosen so that r_n is small in some norm specific to the method.

Hybrid PIMs, for example the Arnoldi-Chebyshev [5] and Arnoldi-Faber [27] methods, require the actual *construction* of the iteration polynomials and hence of the residual polynomials. The latter are usually chosen to have small maximum norm on some compact set Ω, which contains (an estimate of) a set associated with the matrix A, for example the spectrum $\Lambda(A)$, or sets containing $\Lambda(A)$, such as the field of values or a pseudospectrum of A. The set Ω is typically referred to as inclusion set. Using methods of approximation theory, polynomials that are asymptotically optimal on inclusion sets can be constructed. These polynomials decrease to zero on the inclusion set with the fastest possible rate, and hence are well suited to serve as residual polynomials of a hybrid PIM. Discussions of different types of inclusion sets used in hybrid PIMs can be found in [18, 22].

On the other hand, parameter free PIMs, for example the Krylov subspace methods MINRES [24], GMRES [25] and QMR [11] (see e.g. [10] for an overview), automatically yield residual polynomials that have certain minimization properties. With methods of approximation theory, upper bounds on the residual norms of these methods can be derived from the minimal maximum norm of the residual polynomials on some compact set associated with the matrix A. Usually, this kind of *analysis* is based on an inclusion set Ω containing $\Lambda(A)$ [25, Proposition 4], the field of values [4] or a pseudospectrum [21, Theorem 4] of A.

The construction and analysis of PIMs can be treated in a unified framework using conformal maps. Let Ψ denote the exterior mapping function of the inclusion set Ω, i.e. the conformal map from the exterior of the closed unit disk onto the exterior of Ω. Then asymptotically optimal normalized polynomials associated with Ω, for example normalized Faber polynomials, can be derived using Ψ. Furthermore, the maximal rate of convergence to zero on Ω of any sequence of polynomials can be computed from the inverse of Ψ evaluated at the point zero.

Since the residual polynomials are normalized to have value one at the point zero, it generally has to be assumed that $0 \notin \Omega$. This presents a major difficulty. Polynomial approximation on a disconnected set such is not well understood and thus one has to work with Ω being connected and compact. But if the origin is inside the convex hull of $\Lambda(A)$, canonical choices for inclusion sets, such as sets bounded by circles or ellipses, include the origin as well. Some authors have to make strong assumptions on the applicability of their methods or convergence bounds. For example, the hybrid Arnoldi-Faber method by Starke and Varga cannot be applied if $\Lambda(A)$ contains real numbers λ_1 and λ_2 with $\lambda_1 < 0 < \lambda_2$ [27, p. 217], and the convergence analysis of Eiermann [4] is not applicable if the origin is in the field of values of A.

A standard approach to the construction or analysis of PIMs is first to find a feasible inclusion set Ω and then to construct the corresponding exterior mapping function Ψ. For sets bounded by ellipses, Ψ is simply the Joukowsky map and the situation is well understood. Asymptotically optimal polynomials as well as asymptotic convergence factors are known [7, 8]. However, as indicated above, such sets are often not feasible and thus the geometry of Ω gets more complicated. Often the boundary of Ω is not chosen to be analytic and in the computation of Ψ, the Schwarz-Christoffel mapping formula is employed. Particular examples of the use of this formula are the hybrid Arnoldi-Faber method [27] as well as the development by Coleman and Myers [1], who compute a conformal map and Faber polynomials for annular sectors. A disadvantage of the Schwarz-Christoffel approach is that the resulting formulas include quantities which are determined implicitly through integral equations. These can only be solved numerically, which is expensive and sometimes susceptible to round-off errors. We refer to [15, Section 16.10] for prin-