The limit of the spectral radius of nonnegative Toeplitz matrices

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ABSTRACT

A bi-infinite sequence $...t_{-2}, t_{-1}, t_0, t_1, t_2, ...$ of nonnegative numbers defines a sequence of nonnegative Toeplitz matrices $T_n = (t_{ik}), n = 1, ...,$, where $t_{ik} = t_{k-i}, i, k = 1, \ldots, n$. We show that the limit of the spectral radius of T_n, as n tends to infinity, is given by $\inf\{\sigma(\xi) : \xi \in [0, \infty]\}$, where $\sigma(\xi) = \sum_{j \in \mathbf{Z}} t_j \xi^j$. A corresponding result holds in the case of block Toeplitz matrices, where the t_i are $p \times p-$nonnegative matrices and $\sigma(\xi)$ is the spectral radius of $\sum_{j \in \mathbf{Z}} \xi^j t_j$. [1]

1 Introduction

Let $t = \{t_i\}_{i \in \mathbf{Z}}$ be a bi-infinite sequence of complex numbers. It defines a sequence of Toeplitz matrices $T_n = (t_{ik}), n = 1, 2, \ldots$, where $t_{ik} = t_{k-i}, i, k = 1, \ldots, n$. In the case that $t_{-i} = \bar{t}_i, i \in \mathbf{Z}$, i.e. each T_n is a hermitian matrix, the asymptotic distribution of the

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eigenvalues of $T_n, n = 1, ...$, has been studied since the fundamental paper of Szegö [Sze]. See [Hir] for some references on this problem.

Assume now that t is a sequence of nonnegative real numbers. Let $\mu_n(t) = \rho(T_n)$, where $\rho(A)$ is the spectral radius of a square matrix A. As T_n is a principal submatrix of the nonnegative matrix T_{n+1} we deduce that the sequence $\{\mu_n(t)\}_1^\infty$ is monotonously increasing. Let

$$\mu(t) = \lim_{n\to\infty} \mu_n(t) \tag{1.1}$$

and note that $\mu(t) \in [0, \infty]$. We associate with t its symbol

$$\sigma(\xi) = \sum_{-\infty}^{\infty} t_i \xi^i := \lim_{m\to\infty} \sum_{i=-m}^{m} t_i \xi^i, \quad \xi \in [0, \infty]. \tag{1.2}$$

Let

$$\mu_0(t) = \inf_{\xi\in[0,\infty]} \sigma(\xi). \tag{1.3}$$

Again $\mu_0(t) \in [0, \infty]$. The main result of this paper is the equality

$$\mu(t) = \mu_0(t) \tag{1.4}$$

(Theorem 3.4). In the symmetric case, i.e. $t_k = t_{-k}, k = 1, ...$, this result is straightforward, as $\mu_0(t) = \sigma(1) = \max\{|\sigma(\xi)| : |\xi| = 1\}$ [Hir].

We generalize (1.4) to block Toeplitz matrices as follows: Let $t = \{t_i\}_{i\in Z}$ be a sequence of $p \times p$ nonnegative matrices. Then $T_n = (t_{i-j})_{i,j=1}^n$ is an $np \times np$ block Toeplitz matrix and set $\mu_n(t) = \rho(T_n)$. Let $\mu(t)$ be given by (1.1) and let

$$\sigma(\xi) := \lim_{m\to\infty} \rho(\sum_{|i|\le m} \xi^i t_i) \tag{1.5}$$

Define $\mu_0(t)$ by (1.3). Then, under some irreducibility assumptions on the matrices appearing in the sequence t, (1.4) holds.

We summarize briefly the content of our paper. In §2 we show that $\mu(t)$ is a lower bound for the spectral radius of an infinite Toeplitz matrix $T = (t_{i-j})_{i\in\mathbf{Z}}$ with respect to a certain class of absolute monotone norms. We also discuss some needed properties of the lower semi-continuous (l.s.c.) functions. We bring an example of block Toeplitz matrices for which $\mu(t) < \mu_0(t)$. In §3 we prove (1.4) for Toeplitz matrices. In §4 we extend our results to block Toeplitz matrices. In §5 we give another characterization of $\mu(t)$ and an upper bound on $\mu(t)$ for nonnegative tridiagonal block Toeplitz matrices.

2 Preliminaries

Denote the space of all complex, real and nonnegative sequences $t = \{t_i\}_{i\in\mathbf{Z}}$ by $\mathbf{C}^{\mathbf{Z}}, \mathbf{R}^{\mathbf{Z}}$ and $\mathbf{R}_+^{\mathbf{Z}}$ respectively. Let $w = \{w_i\}_{i\in\mathbf{Z}} \in \mathbf{R}_+^{\mathbf{Z}}$ be a sequence with positive coordinates.