Probabilistic Representations and Hyperbound Estimates for Semigroups

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Abstract

In this paper, we study lower order perturbations of a symmetric second order differential operator generating a hypercontractive semigroup. We give a probabilistic representation of the in general not sub-Markovian semigroup associated with the perturbed operator and prove that the perturbed semigroup is also hypercontractive under some exponential integrability conditions on the coefficients.

1 Introduction

Let X be a locally convex Hausdorff topological vector space which is Souslinean and let $(H, <,>_H)$ be a Hilbert space continuously and densely embedded into X . Let μ be a probability measure on $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ stands for the Borel σ-algebra. The starting point of this paper is a second order symmetric differential operator

$$Lu(x) = \frac{1}{2} div(A(x)\nabla u(x)) \tag{1}$$

on $L^2(X, \mu)$ determined by a quadratic form $(\mathcal{E}, D(\mathcal{E}))$, which satisfies a Log-Sobolev inequality. This is equivalent to the associated semigroup being hypercontractive (see e.g. [8]). Consider now a lower order perturbation of the above operator

$$\mathcal{L}_p u(x) = \frac{1}{2} div(A(x)\nabla u(x)) + < B(x), \nabla u(x) > - div(\hat{B}(x)\nabla u(x)) - c(x)u(x). \tag{2}$$

where $B(x) : X \to H, \quad \hat{B}(x) : X \to H$ are proper vector fields and $c(x) : X \to R$ is a scale function.

The problem we are concerned with is when the semigroup T_t associated with the perturbed operator $\mathcal{L}_p$ is also hypercontractive , that is, T_t is a bounded operator from

$L^a(X,\mu)$ to $L^b(X,\mu)$ for $b > a$ and big enough t. Our result states that this is the case if any positive multiple of $|B|, |\hat{B}|, c^- := -inf(c, 0)$ are exponentially integrable. We don't impose any smoothness conditions on the coefficients. This means that $\mathcal{L}_p$ is obtained from L by adding some drifts and very singular potentials $div(\hat{B})$, which are usually distributions.

This kind of perturbation theory has been of substantial interest in the past. If $B = \hat{B} = 0$, the problem was studied in [18] under exponential integrability conditions on the potential term c. The case $\hat{B} = 0$ was treated in [9], with the underline measure μ being Gaussian . Perturbations by changing the measure μ via multiplication by suitable density functions was considered in [4] . For other related works, see, for example , [2], [16].

To achieve our results, we combine probabilistic methods with analytic tools. One of the main results is the probabilistic representation of the semigroup associated with the perturbed operator $\mathcal{L}_p$, which involves both forward and also backward stochastic integration. The finite dimensional version of such a representation appeared first in the paper ([12]) by J.Lunt, T.J.Lyons and T.S.Zhang. Some new machinery must be developed in order to get the infinite dimensional version,see Sections 3,4 and 5. The hypercontractivity of the semigroup is then a consequence of this representation.

The paper is organized as follows: In Section 2 we present our framework. The rigorous definition of the perturbed operator via the associated coercive closed form is contained in Section 3. In Section 4, we present a convergence result of semigroups corresponding to a sequence of coercive closed forms. The result is of its own interest and will also be used subsequently. The forward and backward stochastic integrals are discussed in Section 5. In Section 6, we prove the probabilistic representations of the semigroups and identify the quadratic forms for a class of generalized Feynman-Kac semigroups. The hyperbound estimates of the semigroups are given in Section 7.

2 Framework

Let X be a Souslinean locally convex Hausdorff topological vector space with dual X^*. Let $H \subset X$ be a continuously and densely embedded Hilbert space. By $\mathcal{FC}_b^\infty$ we denote the class of all functions f on X of the form

$$f(x) = \varphi(l_1(x), \ldots, l_n(x)),\ \varphi \in C_b^\infty(R^n),\ l_i \in X^*, \quad x \in X.$$

For $f \in \mathcal{FC}_b^\infty$, let $\nabla f(x)$ denote the gradient of f in H at $x \in X$, i.e.,

$$< \nabla f(x), h >_H = \frac{df(x+\varepsilon h)}{d\varepsilon}|_{\varepsilon=0}, \quad \text{for all } h \in H.$$

where $<,>_H$ stands for the inner product in H.

By $\mathcal{L}(H)$ we denote the collection of all bounded linear operators on H. Let $A(x) : X \to \mathcal{L}(H)$ be a strongly measurable map such that $A(x)$ is symmetric and

$$c_1 I \le A(x) \le c_2 I \tag{3}$$