Coxeter Decompositions of Hyperbolic Tetrahedra.

Felikson A. A.

Introduction

Let P be a polyhedron in a hyperbolic space H^3.

Definition 1. *A polyhedron P is a* **Coxeter polyhedron** *if all dihedral angles of P are the integer parts of π.*

Definition 2. *A polyhedron P admits a* **Coxeter decomposition** *if P can be tiled by N Coxeter polyhedra F_i,...,F_N ($1 < N < \infty$), such that any two tiles F_i and F_j with a common face are symmetric with respect to this face.*

In this paper we classify all Coxeter decompositions of hyperbolic tetrahedra.

Definitions

All polyhedra F_i in a Coxeter decomposition are obviously congruent. We call these polyhedra **fundamental** and denote by F. A plane containing a face of a fundamental polyhedron will be called **mirror** if it contains no face of P.

Remark 1. *A Coxeter decomposition of any polyhedron can be prolonged to the decomposition of the whole hyperbolic space (to define a Coxeter decomposition of the space take definition 2 without the condition $N < \infty$).*

In this paper we deal with polyhedra bounded by the mirrors of a Coxeter decomposition of some polyhedron or the space.

Definition 3. *Fix a Coxeter decomposition of a polyhedron P.*

A **vertex** *A of P is* **fundamental** *if it does not belong to any mirror.*

A **dihedral angle** *of P made up by faces α and β is called* **fundamental** *if $\alpha \cap \beta$ does not belong to any mirror.*

Definition 4. *We say that a point $x \in F_i$* **belongs to a neighborhood** *of a vertex A of P if $A \in F_i$.*

Remark 2. *All the definitions above can be used for polytopes of arbitrary dimension, in particular, for polygons.*

Notation

$N > 1$ is a number of the decomposition tiles F in P.
$Vol(T)$ is a volume of a tetrahedron T.
k-edge is an edge of the fundamental tetrahedron with dihedral angle $\frac{\pi}{k}$.
k-l-m-vertex is a vertex of the fundamental tetrahedron, such that the dihedral angles at the edges intersecting in this vertex are equal to $\frac{\pi}{k}$, $\frac{\pi}{l}$ and $\frac{\pi}{m}$.
A dihedral angle $\frac{k\pi}{q}$ is an angle decomposed into k parts of size $\frac{\pi}{q}$.

The Coxeter tetrahedra are represented by their Coxeter diagrams. Denote any Coxeter hyperbolic tetrahedra by H_i, where i is a number of this tetrahedron in Table 1.

1 Fundamental polyhedron

Lemma 1. *(Obvious).*

1) *Let P be a polyhedron admitting a Coxeter decomposition. Then any polyhedron contained in P admits a Coxeter decomposition too.*

2) *Let P be a polyhedron admitting a Coxeter decomposition. If all dihedral angles of P are fundamental, then P is a Coxeter polyhedron.*

Notation. By $\alpha \cap \beta$ we mean the intersection of the sets α and β in the *inner* part of $\mathbf{H}^3$. Let α be a face of a polyhedron, denote by $\overline{\alpha}$ a plane containing α.

Lemma 2. *Let F be a Coxeter polyhedron.*
Then either F is a tetrahedron or F has two faces α and β such that $\overline{\alpha} \cap \overline{\beta} = \emptyset$.

Proof. It is well known that for any polyhedron without obtuse angles

$$\alpha \cap \beta = \emptyset \quad \Rightarrow \quad \overline{\alpha} \cap \overline{\beta} = \emptyset.$$

Coxeter polyhedra have no obtuse angles. Thus, it is enough to prove that F has faces α and β such that $\alpha \cap \beta = \emptyset$.

Suppose that $\alpha \cap \beta \neq \emptyset$ for any faces α and β of F. Let A be an arbitrary vertex of F. Let α_1,...,α_k be a complete set of faces of F such that $A \in \alpha_i$. Let β be any face of P such that $A \notin \beta$. By assumption, $\alpha_i \cap \beta \neq \emptyset \quad \forall i = 1, ..., k$. Therefore, F has no faces except α_i and β, i.e. F is a pyramid.

Suppose that A is not an ideal vertex of F. It is well known that any bounded Coxeter polyhedron is simple. Thus, the pyramid F is a tetrahedron and everything is proved.

Suppose that A is an ideal vertex of F and the pyramid F is not a tetrahedron. Take two faces (say, α_1 and α_3) having no common edges. Since A does not belong to $\mathbf{H}^3$, $\alpha_1 \cap \alpha_3 = \emptyset$.

□

Lemma 3. *Let P be a tetrahedron. Then F is a tetrahedron.*

Proof. By lemma 2, F has two faces α and β such that $\overline{\alpha} \cap \overline{\beta} = \emptyset$. Take any fundamental polyhedron F_0 in P; let α_0 and β_0 be its disjoint faces. Consider a sequence of fundamental tetrahedra $F_i \in P$, $i \in \mathbf{Z}$, such that $\alpha_i = \alpha_{i+1}$, if i is odd, and $\beta_i = \beta_{i+1}$, if i is even (see Fig 1).

The sequence is finite, since P contains only finitely many fundamental tetrahedra. Let F_k and F_s be the endpoints of the sequence. Then α_k or β_k belongs to some face γ of P. Similarly, α_s or β_s belongs to some face δ of P. Obviously, $\overline{\alpha}_i \cap \overline{\beta}_j = \emptyset \quad \forall i, j$. But γ intersects δ, since P is a tetrahedron. Contradiction.

□

Lemma 4. *Let F be a hyperbolic Coxeter tetrahedron. Then there are only finitely many tetrahedra P admitting a Coxeter decomposition with fundamental tetrahedron F.*

Proof. Any hyperbolic tetrahedron is completely determined by its dihedral angles. The dihedral angles of P should be multiples of the dihedral angles of F. Therefore, there are finitely many admissible sets of dihedral angles of P.

□