On Generalized Fibonacci Groups with Odd Number of Generators *

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Abstract

The geometrical properties of cyclically presented group of Fibonacci type $F(r,m,k)$ and $H(r,m,k)$ are discussed. It is shown that for even r and odd m almost all generalized Fibonacci groups $F(r,m,k)$ can not be fundamental groups of hyperbolic 3–orbifolds of finite volume.

1 Introduction

In the present paper we study geometrical properties of a class of cyclically presented groups of Fibonacci type. In 1965 Conway defined a series of groups $F(2,m)$, $m \geq 3$, with the presentation

$$F(2,m) \; = \; \langle x_1, \ldots, x_m \mid x_i x_{i+1} \; = \; x_{i+2} \quad i = 1, \ldots, m \rangle,$$

where all suffices are by mod m. The natural generalization of these groups was introduced and studied in [8]. The groups $F(r,m)$, $r \geq 2$, $m \geq 3$, with the presentation

$$F(r,m) \; = \; \langle x_1, \ldots, x_m \, | \, x_i x_{i+1} \, \ldots x_{i+r-1} = \; x_{i+r} \quad i = 1, \ldots, m \rangle,$$

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are said to be *Fibonacci groups.* The first question on these groups was: is the group $F(r,m)$ finite? A review of results can be found in [7].

The studding of 3–manifolds uniformized by Fibonacci groups $F(2,m)$ with even number of generators was started by Helling, Kim and Mennicke [5]. We recall that a 3–manifold M is said to be *hyperbolic* if it can be obtained as the quotient-space $M = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is the Lobachevsky (also called hyperbolic) 3–space, and $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is a discrete, torsion–free group of isometries. From [5] the group $F(2,6)$ is the fundamental group of the Hantzsche-Wendt flat 3–manifold, and $F(2,2n)$, $n \geq 4$, is the fundamental group of a closed orientable hyperbolic 3–manifold.

The groups $F(2,m)$ with odd number of generators have completely different geometrical properties. It is easy to see [7, p. 84] that its have a torsion and so, can not be fundametal groups of Euclidean or hyperbolic 3-manifolds. We recall that a *hyperbolic 3–orbifold* $\mathcal{O}$ is the quotient space $\mathbb{H}^3/\Gamma$, where Γ is a discrete group of isometries (in particular, if Γ is torsion–free we get a manifold). It was shown in [9] that if m is odd, then the group $F(2,m)$ can not be the fundamental group of a hyperbolic 3–orbifold of finite volume. However in [6] is proved that $F(2,9)$ is automatic. Moreover, if $r \geq 4$ is even, and $m \geq r$ is odd, then the Fibonacci group $F(r,m)$ can not be the fundamental group of a hyperbolic 3–orbifold of finite volume [12].

In the present paper we consider two series of groups of Fibonacci type. The first series are *generalized Fibonacci groups* $F(r,m,k)$, $r \geq 2$, $k \geq 1$, with the presentation

$$F(r,m,k) = \langle x_1, \ldots, x_m \mid x_i x_{i+1} \cdots x_{i+r-1} = x_{i+r-1+k} \quad i = 1, \ldots, m \rangle,$$

where suffices are by mod m.
The second series are groups $H(r,m,k)$, $r \geq 2$, $k \geq 1$, with the presentation

$$H(r,m,k) = \langle x_1, \ldots, x_m \mid x_i x_{i+1} \cdots x_{i+r-1} = x_{i+r} \ldots x_{i+r+k} \quad i = 1, \ldots, m \rangle,$$

where suffices are by mod m. These groups were introduced in [2]. We have $F(r,m,1) = F(r,m)$ and $H(r,m,1) = F(r,m)$.

Here we generalize the method from [9] and [12] for the groups $F(r,m,k)$. We show that for triples of integers r, m, and k with r – even, m – odd, and m does not divide $r + 2k - 1$ the groups $F(r,m,k)$ can not be the fundamental groups of hyperbolic 3–orbifolds of finite volume. Moreover, we discuss some extremal cases with $m = r + 2k - 1$. In particular our groups are