The Ellis semigroup of the full group of isometries of a locally compact metric space

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Abstract

If the space $S(X)$ of connected components of a locally compact metric space X is compact, with respect to the quotient topology, then the full group of isometries of X is a locally compact, closed subgroup of the group of homeomorphisms of X with respect to the pointwise topology.

Let (X,d) be a locally compact compact metric space and X^X be the set of all maps in X with the pointwise convergence topology. If G is a group of homeomorpisms of X acting on X via the evaluation map then the Ellis semigroup of G, $E(G)$ is the closure of G in X^X.

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If $Iso(X)$ is the full group of isometries of X, we are interested in finding conditions for the space X that ensure that $E(Iso(X))$ is a (*locally compact* topological) group, that is to try to answer the question "when does $E(Iso(X)) = Iso(X)$ hold?" Note that since the pointwise closure of $Iso(X)$ consists of maps that preserve the distance but are not necessarily onto, whenever $E(Iso(X))$ is a group then $E(Iso(X)) = Iso(X)$, so in this case $E(Iso(X))$ is a topological group. If X is compact then every map that preserves the distance is onto, so $E(Iso(X))$ $(= Iso(X))$ is a (compact) topological group. In 1928 D. van Dantzig and B.L. van der Waerden proved that if X is *connected* and locally compact, then $E(Iso(X))$ $(= Iso(X))$ is a *locally compact* topological group [1]. Their methods work only if the space X is connected. The following example shows that the result is not true if X is not connected.

Example. Let $\mathbb{Z}$ be endowed with the Euclidean metric d_E. We modify d_E by letting $d = \min\{1, d_E\}$. Let

$$f_n(x) = \begin{cases} x+1, \text{ if } x \geq 0 \text{ or } x < -n \\ 0, \text{ if } x = -n \\ x, \text{ if } -n < x < 0. \end{cases}$$

It is easy to see that for every $x \in X$ the following equations hold eventually, i.e. for $n >> 0$:

$$f_n(x) = \begin{cases} x+1, \text{ if } x \geq 0 \\ x, \text{ if } x < 0. \end{cases}$$

So, if $f : \mathbb{Z} \to \mathbb{Z}$ is defined correspondingly:

$$f(x) = \begin{cases} x+1, \text{ if } x \geq 0 \\ x, \text{ if } x < 0 \end{cases}$$