MATCHING FOURIER ORBITAL INTEGRALS ON A UNITARY GROUP VIA PERVERSE SHEAVES

Yuval Z. FLICKER

1. The purpose of this note is to show that the proof of the fundamental lemma by Ngô [N2], [N1] in the context of Fourier orbital integrals on the general linear group can be adapted – on making only minor changes – to prove the analogous fundamental lemma in the context of such integrals on the unitary group, as conjectured in [F], for local fields of positive characteristic and for regular elements. That such a simple adaptation is possible is yet another manifestation of the power of the intersection cohomological techniques. This note was written in May-June 1998, and served as a basis for the author's talk at the August 1998 ICM Satellite conference "Algebraic Groups and Related Structures" at Bielefeld. An Appendix below (May 1998) deals separately with the case of the unit element of the Hecke algebra.

Let us recall (part of) the conjecture of [F]. Let F be a local non archimedean field with ring $\mathcal{O}$ of integers and $\operatorname{char} F \neq 2$. Let F'/F be a quadratic unramified field extension and $\mathcal{O}'$ the ring of integers in F'. Denote by $x \mapsto \operatorname{Fr}(x)$ the action of $\operatorname{Gal}(F'/F)$. Write $G' = \operatorname{GL}(n, F')$, and signify by w the matrix $(\delta_{i,n+1-j})$ in G', whose non zero entries are 1 placed on the anti diagonal. For $g = (g_{i,j})$ in G' write $\operatorname{Fr}(g) = (\operatorname{Fr}(g_{i,j}))$, $\tilde{\operatorname{Fr}}(g) = {}^t\operatorname{Fr}(g)^{-1}$, where t signifies transpose, and $\tilde{\operatorname{Fr}}^w(g) = w\,\tilde{\operatorname{Fr}}(g)w$.

Put $\tilde{\mathrm{G}}' = G' \times F'^{\times}$. The unitary group $U = U(F)$ is $\{g \in G'; g = \tilde{\operatorname{Fr}}^w(g)\}$, and the group $\tilde{\mathrm{U}} = \tilde{\mathrm{U}}(F)$ of unitary similitudes is $\{(g,\mu) \in \tilde{\mathrm{G}}'; (g,\mu) = \tilde{\operatorname{Fr}}^w(g,\mu)\}$, where $\tilde{\operatorname{Fr}}^w(g,\mu) = (\mu\,\tilde{\operatorname{Fr}}^w(g), \operatorname{Fr}(\mu))$. Write N for the unipotent upper triangular subgroup of U, and N' for that of G'. Write A for the diagonal subgroup of U, and A' for that of G'. Let $K' = \operatorname{GL}(n, \mathcal{O}')$ be the standard maximal compact subgroup of G', $\tilde{\mathrm{K}}' = K' \times \mathcal{O}'^{\times}$ of $\tilde{\mathrm{G}}'$, $K = K' \cap U$ that of U, $\tilde{\mathrm{K}} = \tilde{\mathrm{U}} \cap \tilde{\mathrm{K}}'$ that of $\tilde{\mathrm{U}}$. We shall use the Haar measure dx on N normalized to assign $K \cap N$ the volume one, as well as the Haar measure on N' which assigns $K' \cap N'$ the volume one.

Let $\mathcal{H}'$ denote the (Hecke) convolution algebra of complex ($\mathbb{C}$)-valued bi-$\tilde{\mathrm{K}}'$-invariant compactly supported functions on $\tilde{\mathrm{G}}' \cap (\mathfrak{gl}(n, \mathcal{O}') \times \mathcal{O}')$, where $\mathfrak{gl}(n)$ signifies the algebra of $n \times n$ matrices. The functions f in $\mathcal{H}'$ are uniquely determined by their constant terms

$$f^B(a,\mu) = \delta'_B(a) \int_{N'} f(ax,\mu)dx,$$

Department of Mathematics, The Ohio State University, 231 W. 18th Ave., Columbus, OH 43210-1174; Email: flicker@math.ohio-state.edu.

Typeset by $\mathcal{AMS}$-TEX

where $a = \operatorname{diag}(a_1, \dots, a_n)$ and

$$\delta'_B(a) = \prod_{i<j} |a_i/a_j|_{F'}^{1/2} (= |(I - \operatorname{Ad}(a))_{\operatorname{Lie}(G'/A')}|^{1/2} \cdot |\operatorname{Jac}_{N'}(x \mapsto a^{-1}xax)|^{-1}).$$

Here $|\cdot|_{F'}$ is the absolute value on F' normalized to assign a generator π of the maximal ideal in $\mathcal{O}'$ the value q^{-2}, where q^2 is the cardinality of $\mathcal{O}'/(\pi)$. Also Lie is for the Lie algebra, and Jac is for the Jacobian.

Let $\mathcal{H}$ denote the convolution algebra of $\mathbb{C}$-valued bi-$\tilde{\mathrm{K}}$-invariant compactly supported functions on $\tilde{\mathrm{U}} \cap (\mathfrak{gl}(n, \mathcal{O}') \times \mathcal{O}')$. The functions ϕ in $\mathcal{H}$ are uniquely determined by their constant terms

$$\phi^B(a, \mu) = \delta_B(a) \int_N \phi(ax, \mu) dx,$$

where $a = \operatorname{diag}(a_1, \dots, a_m, \mu/\operatorname{Fr}(a_m), \dots, \mu/\operatorname{Fr}(a_1))$ if $n = 2m$ or

$$a = \operatorname{diag}(a_1, \dots, a_m, a_{m+1}, \mu/\operatorname{Fr}(a_m), \dots, \mu/\operatorname{Fr}(a_1))$$

if $n = 2m+1$, and then $a_{m+1}\operatorname{Fr}(a_{m+1}) = \mu$; a_i, μ, and $\mu/\operatorname{Fr}(a_i)$ lie in $\mathcal{O}' - \{0\}$ if $\phi^B(a, \mu)$ is non zero. Here $\delta_B(a) = \prod_{i \le m} |a_i|_{F'}^{m-i+v} \cdot \prod_{i \le m} |\mu|_F^{-(m-i)-v}$, where v is $1/2$ when n is even and 1 when n is odd.

There is a base change homomorphism $b : \mathcal{H}' \to \mathcal{H}$, $b(f) = \phi$, which can be defined by: $\phi^B((\tilde{a}, \tilde{\mu})) = \sum f^B(a, \mu)$, the sum ranging over the (a, μ) with $(a, \mu)\tilde{\operatorname{Fr}}^w(a, \mu) = (\tilde{a}, \tilde{\mu})$, and $\phi^B(\tilde{a}, \tilde{\mu}) = 0$ if $(\tilde{a}, \tilde{\mu})$ is not of the form $(a, \mu)\tilde{\operatorname{Fr}}^w(a, \mu)$ for any (a, μ) (in our case where F'/F is unramified this means that $\tilde{\mu}$ has an odd valuation). Note that the norm map

$$N(a, \mu) = (a, \mu)\tilde{\operatorname{Fr}}^w(a, \mu) = (\mu a w \operatorname{Fr}(a)^{-1} w, \mu \operatorname{Fr}(\mu))$$

maps $\tilde{A}' = A' \times F'^{\times}$ onto the group $\{(\mu A', \mu \operatorname{Fr}\mu); \mu \in F'^{\times}\}$, and that $\delta'_B(a) = \delta_B(\mu a \tilde{\operatorname{Fr}}^w(a))$ $(a \in A',\ \mu \in F^{\times})$.

Fix a character $\psi_0 : F \to \mathbb{C}^{\times}$ with conductor $\mathcal{O}$, and note that the above-diagonal entries $x_i = x_{i,i+1}$ of $x \in N$ satisfy $\overline{x_i} = -x_{n-i}$. Then $\psi(x) = \psi_0(\iota^{-1}\sum_{1 \le i < n} x_i)$ is a $\mathbb{C}^{\times}$-valued character on N. For $x \in N'$ put

$$\psi'(x) = \psi(x\tilde{\operatorname{Fr}}^w(x)) (= \psi_0(\iota^{-1} \sum_{1 \le i < n} (x_i - \overline{x}_{n-i})));$$

then ψ' is a $\mathbb{C}^{\times}$-valued character on N'.

The conjecture of [F] relates the Fourier orbital integral

$$I((\mu a, \mu \operatorname{Fr}(\mu)), \phi) = \int_N \int_N \phi(x^{-1} w \mu a x', \mu \operatorname{Fr}(\mu)) \psi(x^{-1}x')\, dx\, dx'$$

$$= \int_N \int_N \phi({}^t x \mu a x', \mu \operatorname{Fr}(\mu)) \psi(x x')\, dx\, dx'$$