Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions

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1 Introduction

In this paper we systematically develop a stochastic calculus for *generalized* Dirichlet forms (cf. [St1]). In particular, we show *Fukushima's decomposition* of additive functionals and an *Itô-type formula* in this framework. The class of generalized Dirichlet forms is much larger than the well-studied class of symmetric and coercive Dirichlet forms as in [FOT 94] resp. [MR 92] and time dependent Dirichlet forms as in [O 92]. It contains examples of an entirely new kind (cf. Section 6, [St1]). Therefore, the results obtained in this paper lead to extensions of the corresponding results in the "classical" theories. In particular the proofs are "locally" completely different (cf. e.g. Theorem 2.3 and Theorem 2.5; though for the reader's convenience we tried to follow the line of argument in [FOT 94] as closely as possible). This difference has several reasons: First of all we *do not assume any sector condition*; in certain cases we have to handle $\mathcal{E}$-quasi-lower-semicontinuous functions instead of $\mathcal{E}$-quasi-continuous functions (cf. e.g. Remark 2.6 (iii)), and finally, the Dirichlet space of the classical situation generalizes to a space $\mathcal{F}$ which is not necessarily stable under composition with normal contractions. In contrast to the classical theory it is not known whether regularity or quasi-regularity alone imply the existence of an associated process. An additional structural assumption on $\mathcal{F}$ is made in $[\mathrm{St1}, p.78, D3]$ (i.e. the existence of a nice intermediate space $\mathcal{Y}$ has to be assumed) in order to construct explicitly an associated m-tight special standard process $\mathbf{M}$.
In addition to the new theoretical results described above we also present new applications. In Section 6 we construct weak solutions to stochastic differential equations in infinite dimensions of the type

$$dX_t \;=\; dW_t + \frac{1}{2}\beta_H^{\mu}(X_t)dt + \overline{\beta}(X_t)dt\ ,\ X_0 = z. \tag{1}$$

Here $(X_t)_{t\geq 0}$ takes values in some real separable Banach space E, $z \in E$, $(W_t)_{t\geq 0}$ is an E-valued Brownian motion, $\overline{\beta}$ is some square integrable vector field on E taking values in a real separable Hilbert space $H \subset E$ and $\beta_H^{\mu} : E \to E$ is the logarithmic derivative of μ associated with H (cf. Subsection 6.1). In the symmetric case, when $\overline{\beta} = 0$, equation (1)

has been studied intensively in [AR 91].
In Subsection 6.2 we give a first application of (1) for more explicit maps β_H^μ and $\overline{\beta}$ using existence results of [BR 95] on invariant probability measures for some given linear operator L, i.e. measures μ solving the equation $\int Lu\, d\mu = 0$ for all finitely based smooth functions u. We also use results of [St2] on the existence of diffusions associated to extensions of such operators. More precisely, in this case we assume E also to be a Hilbert space and that $H \subset E$ densely by a Hilbert-Schmidt map. We then apply (1) with $\overline{\beta} = \frac{1}{2}(B - \beta_H^\mu)$ where $B : E \to E$ is a Borel measurable vector field of the form $B = -id_E + v$, $v : E \to H$, satisfying *(B.1)-(B.3)* of Subsection 6.2. Under these assumptions on B there exists an invariant probability measure μ such that the stochastic differential equation

$$dX_t \;=\; dW_t - \frac{1}{2}X_t dt + \frac{1}{2}v(X_t)dt \ , \ X_0 = z \tag{2}$$

admits a weak solution $\mathbf{M} = (\Omega, (\mathcal{F})_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ for μ-a.e. (even (quasi-)every) $z \in E$. In particular μ is absolutely continuous w.r.t. the Gaussian measure γ on E with Radon-Nikodym derivative φ^2 where φ is in $H^{1,2}(E;\gamma)$, i.e., the Sobolev space over (E, H, γ). Moreover $\beta_H^\mu = -id_E + 2\frac{\nabla\varphi}{\varphi}$. It is known (see [BR 95, Theorem 3.10]) that the generator of $\mathbf{M}$ restricted to the finitely based smooth functions $L = \frac{1}{2}\Delta_H + \frac{1}{2}B \cdot \nabla$ is μ-symmetric if and only if $v = 2\frac{\nabla\varphi}{\varphi}$ or equivalently $B = \beta_H^\mu$. In our general, i.e., non-symmetric situation, $2\frac{\nabla\varphi}{\varphi}$ is the orthogonal projection of v on the closure of the set $\{\nabla u \mid u \in \mathcal{F}\mathcal{C}_b^\infty\}$ in $L^2(E, H; \mu)$. The diffusion $\widehat{\mathbf{M}} = (\widehat{\Omega}, (\widehat{\mathcal{F}})_{t\geq 0}, (\widehat{X}_t)_{t\geq 0}, (\widehat{P}_z)_{z\in E})$ which is in duality to $\mathbf{M}$ w.r.t. μ weakly solves

$$d\widehat{X}_t \;=\; d\widehat{W}_t - \frac{1}{2}\widehat{X}_t dt + 2\frac{\nabla\varphi}{\varphi}(\widehat{X}_t)dt - \frac{1}{2}v(\widehat{X}_t)dt \ , \ \widehat{X}_0 = z \tag{3}$$

for μ-a.e. (even (quasi-)every) $z \in E$ where $(\widehat{W}_t)_{t\geq 0}$ is an E-valued $(\widehat{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance given by the inner product of H. Thus, adding the drifts of (2) and (3) we obtain β_H^μ as in the symmetric case (cf. e.g. [Fö 88]). In Section 6.3 we show that $\mathbf{M}$ also satisfies an Itô-type formula.
Let us now briefly summarize the contents of the remaining Sections 2-5. In Section 2 we introduce the framework and then establish an integral representation theorem for coexcessive functions. As a result we obtain a description of $\mathcal{E}$-exceptional sets in terms of an appropriate class of measures $\widehat{S}_{00}$. This is the key-point for the proof of Theorem 4.5 below. In the symmetric case our class of measures $\widehat{S}_{00}$ is smaller than the corresponding one in [FOT 94, *p.78*]. As a consequence the uniform convergence in Lemma 4.2 can be determined by a weaker semi-norm than in [FOT 94, *Lemma* 5.1.2.]. In Section 3 following [R 70] we associate to every additive functional of $\mathbf{M}$ its corresponding *Revuz measure.* Section 4 is devoted to the *Fukushima-decomposition.* Note that for the proof we *only* assume the coresolvent to be sub-Markovian and strongly continuous on $\mathcal{V}$. *No dual process* is needed. In Remark 4.6(iii) we give conditions for the extension of the decomposition which we use in Subsection 6.3 in an essential way. In Section 5 similarly to [FOT 94], [O 88] we derive an *Itô-type formula* for the transformation of the martingale part of the