HOMOTOPY THEORY OF MODULES AND GORENSTEIN RINGS

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1. Introduction

It is an old observation of Eckmann-Hilton [21], that the homotopy theory of topological spaces has an algebraic analogue in the module category of a ring. Inspiring by the work of Eckmann-Hilton, various authors studied the problem of constructing a homotopy theory in more general algebraic categories. We refer to the works of Heller [18], [20], [19], Huber [23], Kleisli [29], Brown [12], Auslander-Bridger [1] and Quillen [34]. Restricting to the case of a module category, there are two different in general, homotopy theories defined. The injective homotopy which is defined by killing the injective modules and the projective homotopy which is defined by killing the projective modules. Let Λ be an associative ring, and let $\mathrm{Mod}(\Lambda)$ be the category of right $\Lambda-$modules. Using injective homotopy we obtain the stable category $\overline{\mathrm{Mod}}(\Lambda)$ which is always right triangulated, and using projective homotopy we obtain the stable category $\underline{\mathrm{Mod}}(\Lambda)$ which is always left triangulated. The projective and the injective homotopy coincide iff the ring is Quasi-Frobenius (QF-ring for short) and in this case the stable category $\underline{\mathrm{Mod}}(\Lambda) = \overline{\mathrm{Mod}}(\Lambda)$ is a compactly generated triangulated category. The stable module category of a modular group algebra (which is a QF-ring), has been studied by many authors mainly from the representation theoretic point of view. There is recently a big progress in this study, which is developed using machinery from the theory of triangulated categories, in particular Bousfield's localization techniques, see for example [35], [11]. Our main purpose in this paper is to study the stable module categories $\underline{\mathrm{Mod}}(\Lambda)$ and $\overline{\mathrm{Mod}}(\Lambda)$ of a ring Λ from the point of view of modern algebraic homotopy theory. This is possible if the stable module categories are compactly generated, so a theory of Brown Representability can be developed.

The paper is organized as follows.

In Section 2, we study general stable categories with products or coproducts which are induced by homologically finite subcategories in the sense of Auslander-Smalø [2]. In Section 3 we study Brown Representability and its consequences, in a given additive category with coproducts and weak cokernels. Here the notion of a Whitehead subcategory borrowed from topology is of central importance. Brown's Theorem in this setting has interesting applications to (right) triangulated categories, to $t-$structures in the sense of [4] and to locally finitely presented additive categories in the sense of [13]. In particular we show that Bousfield's localization holds in a compactly generated right triangulated category. In Section 4 we prove that a functorially finite subcategory $\mathcal{X}$ of an additive category $\mathcal{C}$, induces a closed model structure on $\mathcal{C}$ in the sense of Quillen [34], with all objects fibrant and cofibrant. Conversely any such closed model structure arises from a functorially finite subcategory. The associated homotopy category is the stable category $\mathcal{C}/\mathcal{X}$ which in many cases has the structure of a pre-triangulated category, i.e. it is left and right triangulated in a compatible way.

Section 5 is devoted to the characterization of rings, such that the stable categories modulo projectives or injectives are "suitable" for doing homotopy theory.

For the injective homotopy this happens iff the ring Λ is right Morita. For the projective homotopy this happens iff the ring Λ is left coherent and right perfect. In both cases the stable categories are compactly generated Abstract Homotopy Categories [12], and Brown Representability Theorem is applicable. We note that our results on injective homotopy generalize some recent results of Jørgensen [25].

In Section 6, inspired from the construction of the stable homotopy category of spectra [32], we study the existence of a stable homotopy category associated to the projective or injective homotopy of a ring Λ. Since the stable module categories are not in general triangulated, it is useful in many cases to replace them by their stabilizations [8], [19], which are triangulated categories, and this can be done in a universal way. We say that a ring Λ has a projective, resp. injective, stable homotopy category if the stabilization of $\underline{\mathrm{Mod}}(\Lambda)$, resp. of $\overline{\mathrm{Mod}}(\Lambda)$, is compactly generated. We prove that in case Λ is right Gorenstein in the sense of [8], and the ring Λ is left coherent and right perfect or right Morita, then such a stable homotopy category exists and can be described as the triangulated stable category of Cohen-Macaulay modules. We close the paper studying when the stable homotopy category is a phantomless or Brown category in the sense of [6], [7].

Throughout this paper we compose morphisms in the diagrammatic order, i.e. the composition of $f : A \to B$, $g : B \to C$ is denoted by $f \circ g$.

2. Stable Categories with Coproducts and Compact Objects

We fix in this section an additive category $\mathcal{C}$ with split idempotents and a full additive subcategory $\mathcal{X} \subseteq \mathcal{C}$ of $\mathcal{C}$ closed under isomorphisms and direct summands. A morphism $f : A \to B$ in $\mathcal{C}$ is called an $\mathcal{X}-$*epic* if the morphism $\mathcal{C}(\mathcal{X}, f) : \mathcal{C}(\mathcal{X}, A) \to \mathcal{C}(\mathcal{X}, B)$ is surjective. We recall from [2] that the subcategory $\mathcal{X}$ is called *contravariantly finite* in $\mathcal{C}$ if for any object $A \in \mathcal{C}$, there exists an $\mathcal{X}-$epic $\chi_A : X_A \to A$ with $X_A \in \mathcal{X}$. The morphism χ_A is called a *right* $\mathcal{X}-$*approximation* of A. The notions of an $\mathcal{X}-$*monic* morphism and of a *covariantly finite* subcategory are defined dually. A subcategory is called *functorially finite* if it is both covariantly and contravariantly finite.

We denote the stable category of $\mathcal{C}$ modulo $\mathcal{X}$ by $\mathcal{C}/\mathcal{X}$. We recall that the objects of $\mathcal{C}/\mathcal{X}$ are the objects of $\mathcal{C}$. If A, B are objects of $\mathcal{C}/\mathcal{X}$, then $\mathcal{C}/\mathcal{X}(A, B) = \mathcal{C}(A, B)/\mathcal{I}_{\mathcal{X}}(A, B)$, where $\mathcal{I}_{\mathcal{X}}(A, B)$ is the subgroup of $\mathcal{C}(A, B)$ consisting of all morphisms factorizing through an object of $\mathcal{X}$. We denote by $\underline{A}$ the object A considered as an object of $\mathcal{C}/\mathcal{X}$ and by $\underline{f}$ the class of the morphism $f : A \to B$ in $\mathcal{C}/\mathcal{X}(A, B)$. Then $\mathcal{C}/\mathcal{X}$ is an additive category and setting $\pi(A) = \underline{A}$ and $\pi(f) = \underline{f}$, we obtain the *projection* functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{X}$.

Proposition 2.1. *Suppose that $\mathcal{C}$ has coproducts (products).*

1. *The functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{X}$ preserves coproducts (products) $\Leftrightarrow$ $\mathcal{X}$ is closed under coproducts (products). In this case $\mathcal{C}/\mathcal{X}$ has coproducts (products).*
2. *If $\mathcal{X}$ is contravariantly finite (covariantly finite), then $\mathcal{X}$ is closed under coproducts (products).*
3. *If $\mathcal{X}$ is closed under coproducts (products), then idempotents split in $\mathcal{C}/\mathcal{X}$.*

Proof. (1) Assume first that $\mathcal{X}$ is closed under coproducts. Let $\{\underline{A}_i; i \in I\}$ be a set of objects in $\mathcal{C}/\mathcal{X}$ and let $\mu_i : A_i \to \oplus A_i$ be the injections into the coproduct in $\mathcal{C}$. Then we have morphisms $\underline{\mu}_i : \underline{A_i} \to \underline{\oplus A_i}$ in $\mathcal{C}/\mathcal{X}$. Let $\underline{f}_i : \underline{A_i} \to \underline{B}$ be morphisms in $\mathcal{C}/\mathcal{X}$. Choose morphisms $f'_i : A_i \to B$ in $\mathcal{C}$, such that $\underline{f'_i} = \underline{f}_i, \forall i \in I$. Then there exists a unique morphism $f' : \oplus A_i \to B$ such that $\mu_i \circ f' = f'_i, \forall i \in I$. Then in $\mathcal{C}/\mathcal{X}$ we have $\underline{\mu}_i \circ \underline{f'} = \underline{f}_i, \forall i \in I$. Let $\underline{h} : \underline{\oplus A_i} \to \underline{B}$ be another morphism in $\mathcal{C}/\mathcal{X}$ such that $\underline{\mu}_i \circ \underline{h} = \underline{f}_i, \forall i \in I$. Then there are morphisms $\kappa_i : A_i \to X_i$ and $\chi_i : X_i \to B$, where $X_i \in \mathcal{X}$, such that $\mu_i \circ f' - \mu_i \circ h' = \kappa_i \circ \chi_i$, where