When is a cellular algebra quasi–hereditary ?

by

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1 Introduction

To a large extent, algebraic representation theory of Lie algebras, algebraic groups and related finite groups deals with finite dimensional algebras which are cellular or quasi–hereditary. Group algebras of symmetric groups and their Hecke algebras are known to be cellular as well as various generalizations (e.g. Brauer algebras, cyclotomic Hecke algebras, Temperley–Lieb algebras, partition algebras). Several of these algebras also have been used in other contexts like topology (invariants of knots or manifolds) or statistical mechanics. Schur algebras associated with semisimple algebraic groups in any characteristic and blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ associated with semisimple complex Lie algebras are cellular as well, but they also satisfy the stronger condition to be quasi–hereditary. A quasi–hereditary structure comes both with desirable numerical properties (decomposition matrices are square matrices, the number of simple modules can be read off from a defining chain of ideals) and with homological structure (finite global dimension, vanishing results on certain cohomology groups, stratification of derived module categories, existence of 'tilting modules' and derived equivalences, possibility to define 'Kazhdan–Lusztig' theory), and also there is a categorical definition (which cannot exist for cellular algebras, see [8]). Many cellular algebras, in particular Brauer algebras and partition algebras, are known to be quasi–hereditary for some choice of parameters and not quasi–hereditary for some other choice (typically 'at zero').

The subject of this note are the following problems arising from this situation.

Problem 1. How to characterize among the cellular algebras the quasi–hereditary ones by a structural property?

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Problem 2. How to characterize among the cellular algebras the quasi–hereditary ones by a numerical property?

Problem 3. Given a cellular algebra with a cell chain of ideals, how to decide whether it is quasi–hereditary?

Problem 4. Determine precisely for which choice of parameters a Brauer algebra or a partition algebra is quasi–hereditary.

Our answer to these problems is contained in the following theorems. More precisely, problem 1 is answered by the equivalence of (a) and (c) in 1.1, problem 2 is answered by the equivalence of (a) and (d), problem 3 is answered by the equivalence of non–(a) and non–(b). Problem 4 is answered by theorems 1.3 and 1.4, which are applications of 1.1.

Theorem 1.1 *Let k be a field and A a cellular k–algebra (with respect to an involution i). Then the following are equivalent:*

(a) Some cell chain of A (with respect to some involution, possibly different from i) is a heredity chain as well, i.e. it makes A into a quasi–hereditary algebra.

(a') There is a cell chain of A (with respect to some involution, possibly different from i) whose length equals the number of isomorphism classes of simple A–modules.

(b) Any cell chain of A (with respect to any involution) is a heredity chain.

(c) A has finite global dimension.

(d) The Cartan matrix of A has determinant one.

Given a k–algebra A, it may be difficult to show that A is not cellular since the definition involves the choice of a basis. Our proof of theorem 1.1 provides us with an invariant of cellular algebras as follows:

Proposition 1.2 *Let k be a field and A a cellular k–algebra. Then the determinant of the Cartan matrix of A is a positive integer.*

We note that our result is in accordance with the socalled 'Cartan determinant conjecture' which states that a finite dimensional algebra of finite global dimension should have Cartan determinant one (not just plus or minus one, as Eilenberg had shown).

Applying the equivalence of (a) and (b) in the theorem, we can solve problem 4, namely we give a precise description for which parameters the Brauer algebras and partition algebras are quasi–hereditary.