On Opposition in Spherical Buildings and Twin Buildings

Peter Abramenko[1*] Hendrik Van Maldeghem[2†‡]

[1] *Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D – 33501 Bielefeld,* abramenk@mathematik.uni-bielefeld.de

[2] *Department of Pure Mathematics and Computer Algebra, University of Ghent, Galglaan 2, B – 9000 Ghent,* hvm@cage.rug.ac.be

Abstract

In this paper, we characterize twin apartments in twin buildings by means of the opposition relation on chambers. We also characterize adjacency of chambers in twin buildings by means of opposition of chambers. As an application, we study maps which satisfy certain conditions related to opposition of chambers, e.g. maps that preserve opposition. Applied to the special case of spherical buildings, all our main results as well as their corollaries are new.

Mathematics Subject Classification 1991: 51E24.
Key words and phrases: twin buildings, spherical buildings, opposition, apartments, generalized polygons, chambers.

1 Introduction and Statement of the Main Results

Twin buildings are natural generalizations of spherical buildings, which play a central role in the modern theories of incidence geometry, group theory, finite geometry, etc. Where

*The first author is supported by the Deutsche Forschungsgemeinschaft through a Heisenberg fellowship

†The second author is a Research Director of the Fund for Scientific Research – Flanders (Belgium)

‡Most of this research was done while the second author was visiting — supported by the Deutsche Forschungsgemeinschaft — the SFB 343 of the University of Bielefeld and gratefully enjoying the kind hospitality of Prof. Dr. H. Abels and the first author

spherical buildings are associated to groups of Lie type, twin buildings arise from Kac-Moody groups. The crucial notion in a twin building is that of *opposition* (of chambers). Roughly speaking, and denoting the set of chambers of a building Ω by $\mathrm{Ch}(\Omega)$, a twin building $\Delta = (\Delta_+, \Delta_-, \delta^*)$ consists of a pair of buildings (Δ_+, Δ_-) of the same type, say determined by the Coxeter System (W, S) (standard notation), and a "codistance" $\delta^* : (\mathrm{Ch}(\Delta_+) \times \mathrm{Ch}(\Delta_-)) \cup (\mathrm{Ch}(\Delta_-) \times \mathrm{Ch}(\Delta_+)) \to W$. The axioms are modelled on the situation in spherical buildings. There, the codistance is in a sense "complementary" to the Weyl distance, in particular the codistance between two chambers is equal to 1 (the identity in W) if and only if the Weyl distance between these two chambers is equal to the longest word. Correspondingly, if Δ is a twin building as before, then two chambers $C, D \in \mathrm{Ch}(\Delta) := \mathrm{Ch}(\Delta_+) \cup \mathrm{Ch}(\Delta_-)$ are called *opposite* precisely when they are in different components and $\delta^*(C, D) = 1$ (there is also a numerical codistance, which is obtained from δ^* by taking the length of the words in (W, S)). Hence spherical buildings can be twinned to themselves and constitute a subclass of the class of twin buildings (cf. TITS [1990], Proposition 1).

The aim of the present paper is to use the opposition of chambers to characterize two fundamental notions in the theory of (twin) buildings: that of "twin apartments" and that of "adjacency of chambers". A twin apartment in a twin building Δ as above is a pair of apartments $\Sigma = (\Sigma_+, \Sigma_-)$, $\Sigma_\epsilon \subseteq \Delta_\epsilon$, $\epsilon \in \{+, -\}$, such that the restriction of the codistance to that pair induces the structure of a (thin) twin building. Applied to the spherical case, every usual apartment can be seen as a twin apartment, unlike the general situation (there are many apartments in Δ_+ which do not have a twin in Δ_- and vice versa). Our characterization of twin apartments uses a very simple property which seems to be unnoticed before, even in the case of spherical buildings. As a result of our theorems, a twin building is completely determined by its set of chambers and the opposition relation (under the additional assumption that every rank 2 residue is spherical).

We would like to emphasize the fact that our main results are also new for the case of spherical buildings. Hence we do not "just generalize" known facts from spherical building theory to twin building theory. In fact, it was just a problem arising from twin buildings, posed to the second author by Bernhard Mühlherr in the disguised form of a question on generalized polygons (which are the rank 2 spherical buildings), that started our investigations. Once the solution was clear for the spherical case, it was a matter of using the right tools to compose a proof that was also valid for the twin case.

Twin buildings were originally invented by Mark Ronan and Jacques Tits. They were introduced in the literature by TITS [1990] (see also TITS [1989]), where many elementary properties are given, but proofs are only sketched or omitted. An introduction and an exposition of many elementary properties with proofs is provided by ABRAMENKO [1996], to which we refer for definitions and notation.

For people only interested in the results restricted to the spherical case, we will phrase our