Tightness of Sobolev capacities in infinite dimensions

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Abstract

We prove tightness of capacities generated by Sobolev classes in locally convex spaces with differentiable measures.

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1 Introduction

The problem of tightness of capacities generated by Sobolev classes of functions over smooth measures arises in measure theory and stochastic analysis. The question of tightness of the classical Sobolev capacities in $\mathbb{R}^n$ was answered positively (see, e.g., [5]), but in the infinite dimensional case specific difficulties appear. The problem under discussion has been considered in many works, in particular, in [2], [3], [1], [8]. The tightness of capacities generated by the classes $W^{p,r}$ is important, in particular, for the construction of the surface measure in infinite dimensional space, see [2], [6], [7], and [1].

In the present work, we consider the Sobolev classes generated by differentiable Radon measures on separable Fréchet spaces.

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2 Sobolev classes

Let (X,μ) be a locally convex space endowed with a Radon probability measure μ, and let H be a separable Hilbert space continuously and densely embedded in X. Here and further on we assume that the measure μ is Fomin differentiable along any vector $h\in H$ and that its logarithmic derivative β^μ (see [2, Definition 5.1.3]) satisfies the estimate

$$\|\beta_h^\mu\|_{L^2(\mu)} \leq const\cdot\|h\|_H \tag{1}$$

for each $h\in H$.

Let E be a separable Hilbert space. We shall denote by $\mathcal{H}(H,E)$ the class of all Hilbert–Schmidt operators from H to E; the Hilbert–Schmidt norm is specified by the formula

$$\|T\|^2_{\mathcal{H}(H,E)} = \sum_{m=1}^{\infty}\|Te_m\|^2_E,$$

where $\{e_m\}$ is an orthonormal basis of H.

The classes of polylinear Hilbert–Schmidt mappings are defined by induction: $\mathcal{H}_n(H,E) = \mathcal{H}(H,\mathcal{H}_{n-1}(H,E))$, $n=2,3,...$, the corresponding norms are defined as follows:

$$\|T\|^2_{\mathcal{H}_n(H,E)} = \sum_{m_1=1}^{\infty}\dots\sum_{m_n=1}^{\infty}\|T(e_{m_1},\dots e_{m_n})\|^2_E.$$

Definition 2.1. *(i) A μ-measurable mapping $F : X \longrightarrow E$ is said to be ray absolutely continuous (RAC) if for any $h\in H$, there exists a mapping $F_h : X \longrightarrow E$ such that $F = F_h$ μ-a.e. and for any $x\in X$ the mapping $t\longmapsto F_h(x+th)$ is absolutely continuous.*

(ii) A μ-measurable mapping $F : X\longrightarrow E$ is called stochastically Gâteaux differentiable (SGD) if there exists a μ-measurable mapping $D_H F : X \longrightarrow \mathcal{H}(H,E)$ such that for any $h\in H$ one has

$$\frac{F(x+th)-F(x)}{t} - D_H F(x)(h) \to 0$$

in measure μ as $t\to 0$.

The stochastic Gâteaux derivative of order $n>1$ is defined inductively according to the formula: $D^n_H F = D_H(D^{n-1}_H F)$.

Definition 2.2. *Let $1<p<\infty$. The Sobolev class $D^{p,1}(\mu,E)$ is defined as follows:*

$$D^{p,1}(\mu,E) = \{f\in L^p(\mu,E) : f \text{ is } RAC, SGD,\ D_H f\in L^p(\mu,\mathcal{H}(H,E))\}.$$

It is endowed with the norm

$$\|f\|_{p,1,E} = \|f\|_{L^p(\mu,E)} + \|D_H f\|_{L^p(\mu,\mathcal{H}(H,E))}$$