STABILITY OF TRAVELING WAVES : DICHOTOMIES AND EIGENVALUE CONDITIONS ON FINITE INTERVALS

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Abstract

If a traveling wave is stable or unstable depends essentially on the spectrum of a differential operator P obtained by linearization. We investigate how spectral properties of the all–line operator P are related to eigenvalues of finite–interval BVPs $Pu(x) = su(x)$, $x_- \leq x \leq x_+, Ru = 0$. Here R is a linear boundary operator, for which we will derive determinant conditions, and the x–interval is assumed to be sufficiently large.

Under suitable assumptions, we show (a) resolvent estimates for large s; (b) if s is in the resolvent of the all–line operator P, then s is also in the resolvent for finite–interval BVPs; (c) eigenvalues of P lead to approximating eigenvalues on finite intervals. These results allow to study the stability question for traveling waves by investigating eigenvalues of finite–interval problems. We give applications to the FitzHugh–Nagumo system with small diffusion and to the complex Ginzburg–Landau equations.

Key words: Traveling waves, stability, exponential dichotomies, eigenvalues, finite boundary value problems
AMS subject classification: 34L05, 34A50, 65L10

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1 Introduction

Under some additional assumptions, the stability of a traveling wave can be characterized in terms of the spectral properties of a linear operator, which we call P: If the spectral values s of P satisfy

$$\mathrm{Re}\, s < 0 \quad \text{or} \quad s = 0$$

and $s = 0$ is a semisimple eigenvalue of P, then the wave is stable. On the other hand, if P has a spectral value s with $\mathrm{Re}\, s > 0$, then the wave is unstable. For results in this direction, see [11, 19, 32], for example.

In this paper we consider waves in one space dimension where P is an operator defined on a subspace V of $L_2 = L_2(\mathbb{R}, \mathbb{C}^m)$, i.e., P acts on functions $v(x)$ defined on the real line $-\infty < x < \infty$. Our aim is to investigate how the desired spectral properties of P are related to spectral properties of *finite–interval* problems,

$$Pv = sv \quad \text{in} \quad J = [x_-, x_+], \quad Rv = 0 \ . \tag{1.1}$$

Here J is a large, but compact interval and R is a suitable linear boundary operator specified below. A motivation for relating the spectral properties of the all-line operator P to those of (1.1) is simply that finite–interval problems (1.1) can be investigated with standard numerical methods whereas a direct computation of the spectrum of P is generally difficult.

A main starting point for our work was the paper by Kreiss, Kreiss, and Peterson [19] where finite–interval eigenvalue problems have also been used to obtain spectral information about an operator on the whole real line. In contrast to [19], we extensively use the theory of exponential dichotomies as we believe that this technical tool makes some of the arguments of [19] more transparent and precise.

We not only consider *spectral properties* of finite–interval problems (1.1), but also obtain sharp estimates for *inhomogeneous BVPs*

$$(sI - P)v = f \quad \text{in} \quad J = [x_-, x_+], \quad Rv = \gamma \ , \tag{1.2}$$

i.e., we estimate the resolvent for finite–interval problems. Our aim is to investigate how the resolvent constant depends on s and J, and what kind of conditions are required for the boundary operator R to obtain such estimates. This is of interest if one replaces a nonlinear evolution problem

$$u_t = f(u, u_x, u_{xx}), \quad x \in \mathbb{R}, \quad t \geq 0 \ , \tag{1.3}$$

on the whole real line by a corresponding problem in a finite interval $x_- \leq x \leq x_+$ with boundary condition $Ru = 0$. For a survey of results using resolvent estimates for nonlinear stability, we refer to [20]. In the present paper we do not discuss the implications of our linear theory for the corresponding nonlinear problems, however.

A traveling wave solution $u(x,t) = U(x - ct)$ of the system (1.3) can be found by solving a second order BVP on the real line for U and by determining the corresponding wave velocity c