Independent sets and 2-factors in edge-chromatic-critical graphs

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Abstract

In 1968, Vizing made the following two conjectures for graphs which are critical with respect to the chromatic index: (1) every critical graph has a 2-factor, and (2) every independent vertex set in a critical graph contains at most half of the vertices.

We prove both conjectures for critical graphs with many edges, and determine upper bounds for the size of independent vertex sets in those graphs.

1 Introduction

In this paper we consider simple graphs. Beside our definitions we use standard graph theoretical notation.

Vizing showed that the chromatic index $\chi'(G)$ of a graph G equals its maximum vertex degree $\Delta(G)$ or $\Delta(G)+1$.

A graph G is Δ-*critical* if it has chromatic index $\Delta+1$ and each proper subgraph of G has smaller chromatic index. If the value of Δ is unimportant we use *critical* graph instead of Δ-critical.

A graph $G = (V, E)$ with more than $\Delta\lfloor\frac{|V|}{2}\rfloor$ edges cannot be colored with Δ colors, since it has too many edges. Such graphs are called *overfull*. Thus it is not the specific structure of the graph which causes its coloring properties. The same holds true for many critical graphs with an odd number of vertices and $\Delta\lfloor\frac{|V|}{2}\rfloor + 1$ edges, and it seems that for fixed n the most critical graphs of order n are overfull, see [1].

It is difficult to find critical graphs which are not overfull. For instance critical graphs of even order cannot be overfull and just recently the first infinite families of Δ-critical graphs of even order, for each $\Delta \geq 3$, were constructed in [2].

There are only few results about the structure of critical graphs and some well known conjectures about their structure do not hold, see [2]. However the following two conjectures of Vizing are still open.

Conjecture 1.1 ([3]) *Every critical graph has a 2-factor.*

Let $\alpha(G)$ be the *independence number* of a graph G, i.e. the maximum size of a set of pairwise non-adjacent vertices in G.

Conjecture 1.2 ([3]) *For every critical graph $G = (V, E)$ holds $\alpha(G) \leq \frac{|V|}{2}$.*

Conjecture 1.1 implies Conjecture 1.2 since a 2-factor already does not allow an independent set with more than half of the vertices. From this also it follows that both conjectures are true for 2-critical graphs.

Let K_0^Δ be obtained from the complete bipartite graph $K_{\Delta,\Delta}$ by subdividing one edge. It is easy to see that K_0^Δ is Δ-critical. We construct an infinite sequence of graphs $K_0^\Delta, K_1^\Delta \ldots$, where K_{i+1}^Δ $(i \geq 1)$ is obtained from K_i^Δ by replacing a vertex of maximum degree by a $K_{\Delta-1,\Delta}$. From results of [2] it follows that K_k^Δ is Δ-critical for each $k \geq 1$. Further $|V(K_k^\Delta)| = 2(\Delta + k\Delta - k) + 1$ and K_k^Δ has an independent set of $\Delta + k\Delta - k$ vertices, and hence Conjecture 1.2 is best possible for every Δ, if true.

The aim of this paper is to prove both conjectures for critical graphs with many edges, and to give upper bounds for the independence number of critical graphs.